# Arithmetic Aspects of Strong $F$-regularity

by

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Abstract

In this dissertation, we investigate the existence and abundance of finite torsors over the regular locus of strongly $F$-regular singularities. We do this by studying how the $F$-signature transforms under this type of finite cover. By restricting our attention to étale torsors, we prove that the local étale fundamental group of a strongly $F$-regular singularity is finite. In fact, we obtain effective bounds on its order in terms of the $F$-signature.

In the general case, we prove that any strongly $F$-regular singularity $X$ admits a finite cover $X^\star \to X$, with $X^\star$ strongly $F$-regular, such that the $X^\star$ has the following property: for all finite group-schemes $G$ whose connected component at the identity is either trigonalizable or nilpotent, we have that every $G$-torsor over the regular locus of $X^\star$ is the restriction of a $G$-torsor over $X^\star$. As a consequence of that proof, we conclude that strongly $F$-regular singularities admit no nontrivial unipotent torsors.

Along the way, we give a new Purity of the Branch Locus result for singularities with $F$-signature larger than 1/2. We also obtain effective bounds on the torsion of divisor class groups of strongly $F$-regular singularities, and globally $F$-regular varieties. Additionally, we prove that canonical covers of strongly $F$-regular (resp. $F$-pure) singularities are strongly $F$-regular (resp. $F$-pure).

For my parents, Eufemia and Víctor.

# Contents

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# Chapter 1

# Introduction

The present dissertation is concerned with the existence and measurement of nontrivial "arithmetic" covers over the regular locus of a strongly $F$-regular singularity. The nontriviality is in the sense that the cover does not come from restricting a cover over the whole spectrum.

The more abundant these nontrivial covers are, the more severe the singularity is considered to be. Thus, the problem we are concerned with is a study on the severity of strong $F$-regularity, from an arithmetic point of view. In a nutshell, our main result is that strong $F$-regularity imposes very strong conditions on the existence of these arithmetic covers. According to this philosophy, this means that strongly $F$-regular singularities are mild relative to this arithmetic sense.

To be precise, by an *arithmetic cover* $V \to U$, we mean a $G$-torsor with respect to a finite group-scheme $G$ over an algebraically closed groundfield $\Bbbk$. Notice that there is a purity result for finite torsors [Mar16]. Thus, the abundance of finite torsors in codimension 1 but not everywhere is a measurement of the severity of a singularity.

It is natural to split our study into two cases, namely, the case where $G$ is étale and the more general case where $G$ may have a nontrivial connected component at the identity.

In the étale case, our main result can be expressed by saying that the local étale fundamental group of a strongly $F$-regular singularity is finite, with order prime to the characteristic and bounded by the reciprocal of the $F$-signature.

In the general case, if $R$ is the germ corresponding to our singularity, we aim to prove the existence of a finite extension of germs $R \subset S$ with the following property: for an important class of finite group-schemes $G$, every finite $G$-torsor over the regular locus of $\operatorname{Spec} S$ is the restriction of a $G$-torsor over $\operatorname{Spec} S$. We prove this result for the class of finite group-schemes whose connected component at the identity is either trigonalizable or nilpotent. Moreover, we prove $R \subset S$ is realized by successive finite extensions

$$R \subset S_1 \subset S_2 \subset \cdots \subset S_t = S$$

where $R_{i-1} = R_i^{G_i}$ for a linearly reductive finite group-scheme $G_i$, and the extension $R_{i-1} \subset R_i$ induces a $G_i$-torsor on the regular loci, this for all $i = 1, ..., t$. That is, for all $i = 1, .., t$ the germ $R_{i-1}$ is the ring of invariants of $R_i$ under the action of some linearly reductive group-scheme $G_i$ such that the corresponding finite cover is a torsor over the regular locus. Our methods are still so effective that we get that the generic degree of $R \subset S$ is at most $1/s(R)$.

In this larger generality, the fact that the order of local étale fundamental group of $X = \operatorname{Spec} R$ is prime to the characteristic is replaced by the statement that any unipotent torsor over the regular locus of $X$ extends across to a torsor over $X$, *i.e.* it is the restriction of a torsor over $X$.

In either of the aforementioned cases, our arguments depend on a transformation rule for the $F$-signature under finite morphism. For example, if $(R, \mathfrak{m}, \mathscr{k}, K) \subset (S, \mathfrak{n}, \mathscr{l}, L)$ is a finite extension of domains equipped with an $S$-linear isomorphism $S \to \operatorname{Hom}_R(S, R)$, say $1 \mapsto T$, such that $T$ is surjective and $T(\mathfrak{n}) \subset \mathfrak{m}$, then the following transformation rule holds

$$[\mathscr{l} : \mathscr{k}] \cdot s(S) = [L : K] \cdot s(R)$$

In this dissertation, we also give some major applications of this transformation rule different from those mentioned above. Among others,

- we give a *Purity of the Branch Locus Theorem* for mild singularities,
- we provide effective bounds for the torsion of the divisor class groups of strongly $F$-regular singularities and also globally $F$-regular varieties,
- and finally, we answer positively a question by K.-i. Watanabe on whether all (Veronese-type) cyclic covers of strongly $F$-regular (resp. $F$-pure) singularities are strongly $F$-regular (resp. $F$-pure).

This dissertation is mostly based on the works [CST16], [Car17].

# Chapter 2

# Finite torsors and strong $F$-regularity

This dissertation is mostly concerned with finite torsors over strongly $F$-regular singularities. Therefore, we shortly survey these two subjects, namely strongly $F$-regular singularities and finite torsors. We dedicate one section to each of them. A third section will summarize the simpler form torsors take over singularities.

## 2.1 Strong $F$-regularity

Let $R$ be a Noetherian $\mathbb{F}_p$-algebra with $p$ a prime number. As a shorthand notation, we will use $q = p^e$ throughout. Then we have the $e$-th Frobenius endomorphism $F^e \colon R \to R$, $r \mapsto r^q$, in such a way that for all $R$-module $M$, we denote by $F^e_* M$ the $R$-module obtained by restriction of scalars via $F^e$.

We denote the $R$-bimodule $\operatorname{Hom}_R(F^e_* R, R)$ by $\mathscr{C}^R_e$. If $\mu_r : R \to R$ denotes multiplication by $r \in R$, then the left $R$-module structure on $\mathscr{C}^R_e$ is given by $r \cdot \varphi = \mu_r \circ \varphi$ (post-multiplication by $r$), whereas the right $R$-module structure by $\varphi \cdot \mu_r = \varphi \circ F^e_* \mu_r$ (pre-multiplication by $r$). However, these two structures are compatible in the following way $r \cdot \varphi = \varphi \cdot r^q$, thus often the left-module structure is determined by the right-module one. Any module theoretic notion on $\mathscr{C}^R_e$ is referred to the left one unless otherwise explicitly stated.

We say that an $\mathbb{F}_p$-algebra is *$F$-finite* if $F^e \colon R \to R$ is (module) finite. This is a very mild condition, for it always holds in "geometric" settings. For example, essentially finite type algebras over a perfect field are $F$-finite. Moreover, the class of $F$-finite $\mathbb{F}_p$-algebras is closed under localization, quotients, adic completions, (strict) Henselizations, and finite extensions. $F$-finite algebras are excellent [Kun76] and always admit a dualizing complex [Gab04].

The next theorem is considered to be the genesis of $F$-singularity theory.

**Theorem 2.1.1** (Kunz Theorem [Kun69a]). *Let $R$ be an $\mathbb{F}_p$-algebra. Then $R$ is regular if and only if $F^e\colon R \to R$ is flat. In case $R$ is $F$-finite, $R$ is regular if and and only if $F^e\colon R \to R$ is locally free. In that case, the rank of $F^e_* R$ at $\mathfrak{p}$ is necessarily $q^\delta$ where $\delta = \dim R_{\mathfrak{p}} + \left[k(\mathfrak{p})^{1/p} : \mathfrak{p}\right]$.*

Therefore, singularity theory in positive characteristic is concerned with deviations of the Frobenius endomorphisms from being flat or free. The study of those deviations is simply called $F$-singularity theory. The following definition is due to M. Hochster and J. Roberts [HR76, HR74].

**Definition 2.1.2** ($F$-purity). An $\mathbb{F}_p$-algebra $R$ is said to be *$F$-pure* if $F^e\colon R \to R$ is pure as a map of $R$-modules for some (then all) $e \in \mathbb{N}$. If $R$ is $F$-finite, $F^e\colon R \to R$ is pure if and only if it is split as an $R$-linear map. This means that $R$ is a free summand of $F^e_* R$. Any surjective map $\varphi \in \mathscr{C}^R_e$ is called an *$F$-splitting*.

*Setup* 2.1.3. In this dissertation, we will work exclusively with Noetherian and $F$-finite algebras and schemes. More generally, if $X$ is a $\mathbb{F}_p$-scheme, we say it is $F$-finite if the Frobenius endomorphism $F^e\colon X \to X$ is finite.

The next type of $F$-singularity is the heart of this dissertation. Its conception and initial properties are due to M. Hochster and C. Huneke.

**Definition 2.1.4** (Strong $F$-regularity [HH89]). An $\mathbb{F}_p$-algebra $R$ is said to be *strongly $F$-regular* if for all $r \in R^\circ$ there exists $e \in \mathbb{N}$ and $\varphi \in \mathscr{C}^R_e$ such that $\varphi(F^e_* r) = 1$. In other words, every $r \in R^\circ$ will generate a free summand of $F^e_* R$ for $e \gg 0$.

**Theorem 2.1.5** (Basic properties of strong $F$-regularity [HH89]). *We have the following properties about strong $F$-regularity.*

*(a) Regularity implies strong $F$-regularity, whereas strong $F$-regularity implies $F$-purity.*

*(b) $R$ is strongly $F$-regular if and only if $R_{\mathfrak{p}}$ is strongly $F$-regular for all $\mathfrak{p} \in \operatorname{Spec} R$.*

*(c) A local ring $(R, \mathfrak{m})$ is strongly $F$-regular if and only if $\hat{R}_{\mathfrak{m}}$ is strongly $F$-regular.*

*(d) A strongly $F$-regular algebra is a product of strongly $F$-regular domains.*

*(e) Strongly $F$-regular domains are normal and Cohen–Macaulay.*

There is an additional aspect of strong $F$-regularity that is of vital importance in this work; namely, strongly $F$-regular singularities are *splinters*. See [Ma88] for an extended discussion on splinters, *c.f.* [Hoc73].

**Definition 2.1.6** (Splinters)**.** A ring $R$ is said to be a *splinter* if it splits off from any finite extension, that is, if $R$ is a direct summand, as an $R$-module, of any finite extension ring.

For a classification of 2-dimensional strongly $F$-regular and $F$-pure singularities, see [Har98b]. On the other hand, it is well known that normal toric rings are strongly $F$-regular.

It is well known that strongly $F$-regular rings in positive characteristic are splinters, from [HH94] and [Hoc77]. The converse statement is one of the main conjectures in commutative algebra. It is known to hold in the excellent $\mathbb{Q}$-Gorenstein case [Sin99, HH94]. Clearly, splinters are $F$-pure.

### 2.1.1 $F$-singularities of pairs

In this subsection, we extend the above definitions to pairs. However, before doing that, we recall some basic notions of divisors on normal schemes.

#### An interlude on divisors

Let $R$ be a normal domain of essentially finite type over a field $\mathcal{k}$ with fraction field $K$ and $X = \operatorname{Spec} R$, or more generally, let $X$ be a normal integral scheme. We denote by $\operatorname{Div}(X)$ the abelian group of *Weil divisors* on $X$, *i.e.* the free abelian group generated by the height-1 prime ideals of $R$. One then has a group homomorphism $K^\times \to \operatorname{Div}(X)$. One says that a (Weil) divisor $D$ is *effective* if its order at every prime is nonnegative; one writes $D \geq 0$. Every nonzero rational function $f \in K^\times$ defines a *principal divisor* $\operatorname{div}_R f$. Its order at $\mathfrak{p}$ is by definition the order of $f$ in the discrete valuation ring $R_\mathfrak{p}$. Two divisors are *linearly equivalent*, say $D_1 \sim D_2$, if $D_1 - D_2$ is principal.

Let $D$ be a divisor on $X$; one defines $\mathscr{O}_X(D) \subset K$ to be the quasi-coherent sheaf associated to the $R$-module

$$R(D) \coloneqq \{f \in K^\times \mid \operatorname{div}_R f + D \geq 0\} \cup \{0\}.$$

In particular, $R \subset R(D)$ if $D$ is effective.

The mapping $D \mapsto \mathscr{O}_X(D)$ gives a bijective correspondence in between Weil divisors on $X$ and reflexive rank 1 subsheaves of $\mathscr{K}$,[1] the constant sheaf of rational functions. One says $D$ is *Cartier* if $\mathscr{O}_X(D)$ is invertible. We also consider the $\mathbb{Q}$-space of $\mathbb{Q}$-divisors $\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. The twist $R(\Delta)$ of $R$ by a $\mathbb{Q}$-divisor $\Delta$ works *verbatim* as in the integral case. A $\mathbb{Q}$-divisor $\Delta$ is called $\mathbb{Q}$-*Cartier* if there is $n \in \mathbb{N}$ such that $n\Delta$ is a Cartier integral divisor. A *canonical divisor* on $X$ is by definition a Weil divisor $K_R = K_X$ on $X$ so that $\mathscr{O}_X(K_X)$ is isomorphic to the *dualizing sheaf*[2] $\omega_X$.

Our next goal is to use the formalism of divisors to understand sections of finite extensions $R \subset S$ of normal domains. Let $f \colon Y \to X$ be the corresponding morphisms of schemes. Choose canonical divisors $K_R$ and $K_S$ for $R$ and $S$, respectively. We have that the *relative canonical module* $\omega_{S/R} \coloneqq \mathrm{Hom}_R(f_*S, R)$ is a reflexive rank 1 $S$-module; we want to realize it as $S(D)$ for some divisor $D$.

For this, we recall first the notion of *pullback of divisors under finite morphisms*. Let $\Delta$ be a $\mathbb{Q}$-divisor on $X$. One defines $f^*\Delta$ on $Y$ as follows. The order of $f^*\Delta$ at a height-1 prime ideal of $S$, say $\mathfrak{q}$, is the product $e \cdot a$, where $a$ is the order of $\Delta$ at $\mathfrak{p} = f(\mathfrak{q}) = \mathfrak{q} \cap R$ and $e$ is the *ramification index of $f$ along* $\mathfrak{q}$.[3]

**Example 2.1.7** (Pullback along Frobenius)**.** Let $F^e \colon R \to R$ be the $e$-th Frobenius endomorphism, then $(F^e)^*\Delta = q\Delta$ for all $\mathbb{Q}$-divisor $\Delta$. This is simply because the ramification index of $F^e$ along any height-1 prime is $q$.

Next, we observe the following,

$$\begin{aligned}\mathrm{Hom}_R(f_*S, R) \cong \mathrm{Hom}_R\big(f_*S \otimes_R R(K_R), R(K_R)\big) &\cong \mathrm{Hom}_R\big(f_*S(f^*K_R), R(K_R)\big)\\ &\cong f_*\,\mathrm{Hom}_S\big(S(f^*K_R), S(K_S)\big)\\ &\cong f_*\,\mathrm{Hom}_S\big(S, S(K_S - f^*K_R)\big)\\ &\cong f_*S(K_S - f^*K_R)\end{aligned}$$

[1] A coherent sheaf/module $M$ is *reflexive* if the natural map $M \to M^{\vee\vee}$ is an isomorphism. On a normal scheme, this is equivalent to the second Serre's condition $\mathbf{S}_2$ on the local depths $\operatorname{depth} M_x \geq \min\{2, \dim \mathscr{O}_{X,x}\}$ for all $x \in X$; see [Har94, Theorem 1.9] for further details. By rank, we mean generic rank.

[2] Since $X$ is assumed $F$-finite, it has a dualizing complex $\omega_X^\bullet$ [Gab04]. In case $X$ is Cohen–Macaulay, $\omega_X^\bullet$ is concentrated in degree $\dim X$ and one calls that sheaf the dualizing sheaf of $X$. In general, one defines the dualizing sheaf on $X$ as the unique reflexive sheaf on $X$ agreeing with the dualizing sheaf on the Cohen–Macaulay locus.

[3] Namely, $e$ is the order of any uniformizer of $R_{\mathfrak{p}}$ in the DVR $S_{\mathfrak{q}}$.

where we used [Har77, Chapter II, Exercises 5.1 and 5.2], which are valid, up to reflexification, on a normal scheme $X$ with $\mathscr{E}$ reflexive (rather than just locally free of finite rank). For example, we used the projection formula to say $\big(f_*S \otimes_R R(K_R)\big)^{\vee\vee} \cong f_*S(f^*K_R)$ and therefore, $\mathrm{Hom}_R\big(f_*S \otimes_R R(K_R), R(K_R)\big) \cong \mathrm{Hom}_R\big(f_*S(f^*K_R), R(K_R)\big)$ in the second isomorphism. We also used duality for finite morphisms [Har77, Chapter III, Exercise 6.10] in the third isomorphism.

This gives an isomorphism of $S$-modules $\omega_{S/R} \cong S(K_S - f^*K_R)$. Thus, to every nonzero $R$-linear map $T\colon f_*S \to R$, there corresponds an effective divisors on $Y$, say $D_T$, linearly equivalent to $K_S - f^*K_R =: K_{S/R}$, the relative canonical divisor. Moreover, two maps $T_1, T_2\colon f_*S \to R$ correspond to the same divisor if and only if $T_1 = T_2 \cdot u$ for some unit $u \in S^\times$. In other words, there is a bijection

$$\omega_{S/R} \big/ S^\times \xrightarrow{\cong} \big\{D \in \mathrm{Div}(X) \;\big|\; D \geq 0 \text{ and } D \sim K_{S/R}\big\}, \quad T \mapsto D_T.$$

In practice, one may compute $D_T$ as follows. Let $\mathfrak{q}$ be a height-1 prime ideal of $S$; to compute the order or coefficient of $D_T$ at $\mathfrak{q}$, we consider $\mathfrak{p} = f(\mathfrak{q}) = \mathfrak{q} \cap R$. Then $R_\mathfrak{p} \subset S_\mathfrak{p}$ is a finite extension with $R_\mathfrak{p}$ a DVR and $S_\mathfrak{p}$ a semi-local Dedekind domain, then a PID and Gorenstein. In particular, $\mathrm{Hom}_{R_\mathfrak{p}}(S_\mathfrak{p}, R_\mathfrak{p})$ is a free $S_\mathfrak{p}$-module of rank 1. After a choice of a free generator, then any map in $\mathrm{Hom}_{R_\mathfrak{p}}(S_\mathfrak{p}, R_\mathfrak{p})$ corresponds to a unique $s \in S_\mathfrak{p}$. Although $s$ depends on the choice of the generators, $\mathrm{div}_{S_\mathfrak{p}} s$ certainly does not. This is in fact what the formalism of divisors really does: maps in $\mathrm{Hom}_{R_\mathfrak{p}}(S_\mathfrak{p}, R_\mathfrak{p})$ are in correspondence with principal divisors $\mathrm{div}_{S_\mathfrak{p}} s$ on $\mathrm{Spec}\, S_\mathfrak{p}$. Now, let $\mathrm{div}_{S_\mathfrak{p}} s_T$ be the divisor on $\mathrm{Spec}\, S_\mathfrak{p}$ corresponding to $T_\mathfrak{p} = T \otimes_R R_\mathfrak{p}$. Then, the coefficient of $D_T$ at $\mathfrak{q}$ is the coefficient of $\mathrm{div}_{S_\mathfrak{p}} s_T$ at $\mathfrak{q}S_\mathfrak{p}$.

**Example 2.1.8** (Trace and Ramification)**.** Suppose $R \subset S$ is generically separable, and let $K \subset L$ be the corresponding generic extension. It is not difficult to see that the trace map $\mathrm{Tr}_{L/K}\colon L \to K$ restricts to a map $\mathrm{Tr}_{S/R}\colon S \to R$ on the integral level.[4] We have that $\mathrm{Tr}_{S/R}$ corresponds to the *ramification divisor* Ram.[5] A proof and further details can be found in [ST14, Proposition 4.8]. However, showing this reduces, after

[4]Indeed, if $s \in S \subset L$, then its minimal polynomial over $K$ has coefficients in $R$; see [AM69, Proposition 5.15]. However, $\mathrm{Tr}_{L/K}(s)$ is a $\mathbb{Z}$-multiple of one of those coefficients.

[5]The ramification divisor is defined as follows. Since $R \subset S$ is generically separable, the module of differentials $\Omega_{S/R}$ is generically zero, so a torsion $S$-module. Then, one defines the order of Ram at a height-1 prime ideal $\mathfrak{q}$ to be the length of $(\Omega_{S/R})_\mathfrak{q}$ as an $S_\mathfrak{q}$-module. Equivalently, if $\mathfrak{q} \cap R = \mathfrak{p}$, then the order of Ram at $\mathfrak{q}$ is the same as the order of $\mathrm{div}_{S_\mathfrak{p}} s$ at $\mathfrak{q}S_\mathfrak{p}$ for $(s) = \mathrm{Fitt}_{S_\mathfrak{p}} \Omega_{S_\mathfrak{p}/R_\mathfrak{p}}$.

localizing at a height-1 ideal in Spec $R$, to proving that the Dedekind's different ideal equals the Kähler different ideal. For this last statement, see [Kun86, Proposition 10.17].

*Remark* 2.1.9. Additionally, the same computation in [Har77, Chapter IV, Proposition 2.2] shows that the order of Ram at $\mathfrak{q}$ is at least $e_{\mathfrak{q}} - 1$, where $e_{\mathfrak{q}}$ is the ramification index of $f : \operatorname{Spec} S \to \operatorname{Spec} R$ along $\mathfrak{q}$. Moreover, equality holds if and only if the extension of DVR's $R_{f(\mathfrak{q})} \subset S_{\mathfrak{q}}$ is *tame*, meaning that $e_{\mathfrak{q}}$ is prime-to-$p$ in case $\operatorname{char}(\mathscr{k}) = p > 0$ and the extension of residue fields is separable.

Once we have chosen a section $T \in \omega_{S/R}$, we get an isomorphism $\omega_{S/R} \cong S(D_T)$. This isomorphism can be made very explicit. Indeed, let $T_0 \colon L \to K$ be the localization of $T$ at the generic point. So that we have a commutative square

$$\begin{array}{ccc} L & \xrightarrow{T_0} & K \\ \subset\uparrow & & \uparrow\subset \\ S & \xrightarrow{T} & R \end{array}$$

By construction, $S \subset S(D_T) \subset L$ is the largest $S$-submodule of $L$ whose image under $T_0$ is contained in $R$. So it makes sense to say that $T$ extends to a map in $\operatorname{Hom}_R\big(f_*S(D_T), R\big)$ and this is its maximal extension to a $S$-submodule of $L$. Moreover, the following map is an isomorphism of $S$-modules

$$S(D_T) \to \omega_{S/R}, \quad s \mapsto T(s \cdot -).$$

Following [Sch09], since $R$ is $F$-finite,[6] the previous discussion in particular applies to the most important finite extension in this dissertation, namely $F^e \colon R \to R$. So that we have an isomorphism of right $R$-modules

$$\operatorname{Hom}_R(F^e_* R, R) \cong R\big(K_R - (F^e)^* K_R\big) \cong R\big((1-q)K_R\big)$$

and a correspondence

$$\mathscr{C}^R_e \big/ R^{\times} \xrightarrow{\cong} \big\{D \in \operatorname{Div}(X) \bigm| D \geq 0 \text{ and } D + (q-1)K_R \sim 0\big\}, \quad \varphi \mapsto D_\varphi$$

where the action of $R^\times$ on $\mathscr{C}^R_e$ is on the right (pre-multiplication). Normalizing $D_\varphi$, we define the $\mathbb{Q}$-divisor

$$\Delta_\varphi := \frac{1}{q-1} \cdot D_\varphi.$$

[6] Now assuming $\mathscr{k}$ has positive characteristic $p$.

Then we have the bijection [Sch09]

$$\mathscr{C}_e^R \Big/ {R^\times} \xrightarrow{\cong} \bigl\{\Delta \in \operatorname{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \bigm| \Delta \geq 0 \text{ and } (q-1)(\Delta + K_R) \sim 0\bigr\}, \quad \varphi \mapsto \Delta_\varphi.$$

We close this interlude by stating the "functoriality" property of $T \mapsto D_T$.

**Proposition 2.1.10.** *Let $A \subset B \subset C$ finite inclusions of normal domains with corresponding morphisms of schemes $Z \xrightarrow{g} Y \xrightarrow{f} X$. If $U \in \omega_{C/B} = \operatorname{Hom}_B(g_*C, B)$ and $T \in \omega_{B/A} = \operatorname{Hom}_A(f_*B, A)$, then*

$$D_{T \circ U} = D_U + g^* D_T$$

*on $Z$.*

*Proof.* This equality can be proven after localizing at a height-1 prime ideal of $A$. Concretely, this means that we can assume $R$ to be a DVR and $B$ and $C$ semi-local Dedekind domains, hence PID and Gorenstein. Therefore, we may choose free generators $\Phi$, $\Psi$ for $\omega_{B/A}$ and $\omega_{C/B}$ as $B$-modules and $C$-modules, respectively.

Now, we claim $\Phi \circ \Psi$ is a free generator of $\omega_{C/A}$. Although this can been seen as abstract nonsense about upper-shriek functors and duality,[7] we give a direct proof for sake of completeness. Let $\gamma\colon C \to A$ in $\omega_{C/A}$. It defines a $C$-linear map $\eta\colon C \to \operatorname{Hom}_A(B, A)$ given by $c \mapsto \bigl(b \mapsto \gamma(bc)\bigr)$. Moreover, $\eta(c)(1) = \gamma(c)$ for all $c \in C$. On the other hand, we have a $B$-isomorphism $B \to \operatorname{Hom}_A(B, A)$, $1 \mapsto \Phi$; let $\beta\colon \operatorname{Hom}_A(B, A) \to B$ be the inverse isomorphism. It has the property $\Phi\bigl(\beta(\phi)\bigr) = \phi(1)$ for all $\phi \in \operatorname{Hom}_A(B, A)$.[8] By plugging in $\phi = \eta(c)$, we conclude that the composition $\alpha = \beta \circ \eta\colon C \to B$ satisfies that $\Phi \circ \alpha = \gamma$. But $\alpha = \Psi(c \cdot -)$ for a unique $c \in C$, then $\gamma = (\Phi \circ \Psi)(c \cdot -)$, as claimed.

Next, let $D_U = \operatorname{div}_C u$ and $D_T = \operatorname{div}_B t$. Then, $U = \Psi(u \cdot -)$ and $T = \Phi(t \cdot -)$. Therefore, $T \circ U = (\Phi \circ \Psi)(ut \cdot -)$ and

$$D_{T \circ U} = \operatorname{div}_C(ut) = \operatorname{div}_C u + \operatorname{div}_C t = \operatorname{div}_C u + g^* \operatorname{div}_B t = D_U + g^* D_T.$$

This proves the proposition. ☕

[7] Namely, there are natural isomorphisms $(f \circ g)^! \to g^! \circ f^!$ [Har66].

[8] For by definition $\phi(-) = T\bigl(\beta(\phi) \cdot -\bigr)$.

**Strong $F$-regularity of pairs**

Let $(R, \Delta)$ be a *pair*, *i.e.* $R$ is a normal domain and $\Delta$ an effective $\mathbb{Q}$-divisor on $\operatorname{Spec} R$. Then we consider $\mathscr{C}_e^{\Delta} \subset \mathscr{C}_e^R$ to be the $R$-subbimodule of maps $\varphi\colon F_*^e R \to R$ such that $\Delta_\varphi \geq \Delta$, *i.e.* $D_\varphi \geq (q-1)\Delta$. But since $D_\varphi$ is integral, we have that this "inequality" is equivalent to $D_\varphi \geq \lceil (q-1)\Delta \rceil$. Therefore, by the interlude we had on divisors and finite morphisms, this corresponds to the set of maps $\varphi\colon F_*^e R \to R$ admitting a (necessarily) unique extension to a map $F_*^e R\big(\lceil (q-1)\Delta \rceil\big) \to R$. In other words, $\mathscr{C}_e^{\Delta}$ is just the restrictions of maps in $\operatorname{Hom}_R\Big(F_*^e R\big(\lceil (q-1)\Delta \rceil\big), R\Big)$ to $F_*^e R$ via the inclusion $R \subset R\big(\lceil (q-1)\Delta \rceil\big)$. By abuse of notation, we simply write

$$\mathscr{C}_e^{\Delta} = \operatorname{Hom}_R\Big(F_*^e R\big(\lceil (q-1)\Delta \rceil\big), R\Big) \subset \operatorname{Hom}_R(F_*^e R, R) = \mathscr{C}_e^R$$

having always in mind the above interpretations.

In this way, the concepts of $F$-purity and $F$-regularity generalize to pairs as follows (approach due to K. Schwede).

**Definition 2.1.11** ($F$-singularity of pairs [Sch10b])**.** Let $(R, \Delta)$ be a pair. One says $(R, \Delta)$ is *(sharply) $F$-pure* if there exists $\varphi \in \mathscr{C}_e^{\Delta}$ such that the composition

$$R \xrightarrow{1 \mapsto F_*^e 1} F_*^e R\big(\lceil (q-1)\Delta \rceil\big) \xrightarrow{\varphi} R$$

is the identity, *i.e.* $\varphi(F_*^e 1) = 1$. The pair $(R, \Delta)$ is said to be *strongly $F$-regular* if for all $0 \neq r \in R$, there exists $\varphi \in \mathscr{C}_e^{\Delta}$ such that the composition

$$R \xrightarrow{1 \mapsto F_*^e r} F_*^e R\big(\lceil (q-1)\Delta \rceil\big) \xrightarrow{\varphi} R$$

is the identity, *i.e.* $\varphi(F_*^e r) = 1$.

Of course, the strong $F$-regularity (resp. $F$-purity) of a pair $(R, \Delta)$ implies the strong $F$-regularity (resp. $F$-purity) of $R$.

### 2.1.2 The $F$-signature

There is a numerical characterization of strong $F$-regularity, which we proceed to describe in this subsection. To every local pair[9] $(R, \mathfrak{m}, \mathscr{k}; \Delta)$, one associates a real number $s(R, \Delta) \in [0, 1]$, the *$F$-signature* of the pair, with the following properties.

[9]In this case, we assume $R$ is normal only if $\Delta \neq 0$, for normality is imposed to have a well-behaved theory of divisors but is not required to define the $F$-signature.

**Theorem 2.1.12** (Properties of the $F$-signature)**.** *We have the following properties*

*(a)* $s(R,\Delta) = 1$ *if and only if* $R$ *is regular and* $\Delta = 0$ *[HL02, Corollary 16], [BST13b].*[10]

*(b)* $s(R,\Delta) = 0$ *if and only if* $(R,\Delta)$ *is not strongly* $F$*-regular [AL03, Theorem 0.2], [BST12, Theorem 3.18].*

*(c) The function* $\mathfrak{p} \mapsto s(R_{\mathfrak{q}},\Delta)$ *is lower-semicontinous on* $\operatorname{Spec} R$ *[Pol18].*

The $F$-signature for local rings was formally defined by C. Huneke and G. Leuschke [HL02], although it was implicit in [SdB97]. Nonetheless, the proof of its existence only came years later by the work of K. Tucker [Tuc12]. The formulation for pairs (and for general Cartier algebras) was given in [BST12] by M. Blickle, K. Schwede, and K. Tucker. We strongly recommend [PT18] for further details and simplified proofs.

It is worth remarking that the $F$-signature is often thought of as a volume attached to the singularity or pair. This intuition is for example supported by the work of M. Von Korff [Von12] where the $F$-signatures of toric singularities and pairs are realized as the volumes of polytopes; compare with the earlier results [BST12, Theorem 4.20], [iWiY04, Sin05].

To define the $F$-signature $s(R,\Delta)$ of a pair, we need to define first the *$F$-splitting numbers* $a_e(R,\Delta)$. There are at least three equivalent ways to define these numbers. The first and more intuitive way is to say it is the largest rank $a$ of a free $R$-module $R^{\oplus a}$ that $F^e_* R$ maps onto such that all the $a$ projections

$$F^e_* R \to R^{\oplus a} \to R$$

are in $\mathscr{C}^{\Delta}_e$. Equivalently, it is the maximum rank of a free direct summand of $F^e_* R$ as an $R$-module whose splitting maps $F^e_* R \to R$ are all in $\mathscr{C}^{\Delta}_e$. Therefore, it can be computed as follows

$$a^{R,\Delta}_e = \lambda_R\Big(\mathscr{C}^{\Delta}_e \Big/ \big(\mathscr{C}^{\Delta}_e\big)^{\mathrm{ns}}\Big) = \lambda_R\Big(F^e_* R \Big/ F^e_* I^{\Delta}_e\Big) = q^{\alpha(R)} \cdot \lambda_R\big(R \,/\, I^{\Delta}_e\big)$$

where $\big(\mathscr{C}^{\Delta}_e\big)^{\mathrm{ns}} \subset \mathscr{C}^{\Delta}_e$ is the subbimodule of nonsurjective maps, *i.e.*

$$\big(\mathscr{C}^{\Delta}_e\big)^{\mathrm{ns}} = \operatorname{Hom}_R\Big(F^e_* R\big(\lceil (q-1)\Delta \rceil\big), R\Big) \cap \operatorname{Hom}_R(F^e_* R, \mathfrak{m})$$

[10] Indeed, if $s(R,\Delta) = 0$, then $R$ is regular, forcing $\Delta$ to be $\mathbb{Q}$-Cartier, say $\Delta = t \cdot \operatorname{div}_R f$ for some $f \in R$ and rational $t \in \mathbb{Q}$. Then $s(R,\Delta) = s(R, f^t)$ as treated in [BST13b]. If $s(R, f^t) = 0$, then it follows $(f) = R$ by [BST13b, Theorem 4.6].

$I_e^{\Delta}$ is the ideal of $R$ given by

$$I_e^{\Delta} = \left\{ r \in R \;\middle|\; \varphi(F_*^e r) \in \mathfrak{m} \text{ for all } \varphi \in \mathscr{C}_e^{\Delta} \right\}$$

and $p^{\alpha(R)} := \left[ \mathscr{k}^{1/p} : \mathscr{k} \right]$.

Notice that the largest $a_e(R,\Delta)$ can be is the generic rank of $F_*^e R$ as an $R$-module. That is,

$$0 \leq a_e(R,\Delta) \leq q^{\delta}$$

where $\delta = \dim R + \alpha(R)$. In fact, $q^{\delta}$ is no other but the value $a_e(R,\Delta)$ takes when $R$ is regular and $\Delta = 0$. In this fashion, the ratio

$$0 \leq \frac{a_e(R,\Delta)}{q^{\delta}} \leq 1$$

is a measurement of the severity of the singularity $(R,\Delta)$. As a matter of fact, the limit of these ratios as $e$ goes to infinity exists [Tuc12, BST12, PT18]. That limit is by definition the $F$-signature $s(R,\Delta)$ of the pair $(R,\Delta)$. However, it is unknown whether or not the $F$-signature is rational. Nevertheless, a conjecture of $P$. Monsky [Mon08, Conjecture 1.5] would imply the existence of a singularity with irrational, but algebraic, $F$-signature. In fact, this conjecture would imply that the $F$-signature of $R = \mathbb{F}_2[\![x,y,z,u,v]\!] \,/\, \left(uv + x^3 + y^3 + xyz\right)$ is $\frac{2}{3} - \frac{5}{14\sqrt{7}}$; see [Tuc12, Proposition 4.22] for further details.

*Remark* 2.1.13. We point out that in the present work, we will compute the $F$-splitting numbers $a_e(R,\Delta)$ as the $R$-lengths of $\mathscr{C}_e^{\Delta} \,/\, \left(\mathscr{C}_e^{\Delta}\right)^{\mathrm{ns}}$. We will not compute them as the $R$-colengths of the ideal $I_e^{\Delta}$ in what follows. This is a major difference between the approach in this dissertation and the one in [CST16].

The following property will be important later on. It in particular implies that strong $F$-regularity is invariant under (strict) Henselizations.

**Proposition 2.1.14** ([Yao06, CST17])**.** *Let $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a local faithfully flat homomorphisms, and let $f\colon \operatorname{Spec} S \to \operatorname{Spec} R$ be the corresponding morphism of schemes. Let $\Delta$ be a $\mathbb{Q}$-divisor on $\operatorname{Spec} R$ (if $R$ is normal). If the closed fiber of $f$ is regular, then $s(S, f^*\Delta) = s(R,\Delta)$. In particular, the $F$-signature is invariant under (strict) Henselizations and $\mathfrak{m}$-adic completions (then so is strong $F$-regularity).*

### 2.1.3 Connection with the Minimal Model Program

In this section, we briefly recall the relationship between $F$-singularities and the singularities in the Minimal Model Program (MMP). Our main goal with this is to illustrate that $F$-singularity theory interacts with the field of birational geometry. We invite the reader to glimpse any of the excellent surveys [Pat16, PST17, SZ15] for additional information on this rich, deep connection.

Let $(R, \Delta)$ be a pair[11] over a field $\mathscr{k}$ of any characteristic. One says that $(R, \Delta)$ has *Kawamata log terminal (KLT) singularities (resp. log canonical singularities)* if $K_X + \Delta$ is $\mathbb{Q}$-Cartier and for all normal $\mathscr{k}$-variety $Y$ and all proper birational morphism $f\colon Y \to X$, we have that the coefficients/orders of $f^*(K_X + \Delta) - K_Y$ are strictly less than 1 (resp. less than or equal to 1). Then we have the following fundamental theorem by N. Hara and K.-i. Watanabe.

**Theorem 2.1.15** ([HW02]). *Suppose $\mathscr{k}$ has positive characteristic. If $(R, \Delta)$ is strongly $F$-regular (resp. $F$-pure), then $(R, \Delta)$ has KLT (resp. log canonical) singularities.*

One also has a partial converse statement in the context of reductions to positive characteristic. More precisely, let $(R, \Delta)$ be a pair over $\mathbb{C}$. Let $A$ be a finite $\mathbb{Z}$-algebra over which both $R$ and $\Delta$ are defined, and let $(R_A, \Delta_A)$ be the corresponding model of $(R, \Delta)$ over $A$. We think of $(\operatorname{Spec} R_A, \Delta_A) \to \operatorname{Spec} A$ as a family of models of the original pair $(\operatorname{Spec} R, \Delta)$, where fibers at the closed points of $\operatorname{Spec} A$ are positive characteristic models whereas fiber is a characteristic 0 model. Colloquially, one says that one spreads $(R, \Delta)$ out to positive characteristic. One defines $(R, \Delta)$ to have *strongly $F$-regular type (resp. $F$-pure type)* if the closed fibers of $(\operatorname{Spec} R_A, \Delta_A) \to \operatorname{Spec} A$ are strongly $F$-regular (resp. $F$-pure) for a Zariski dense of closed points in $\operatorname{Spec} A$.[12] This definition is independent of the choice of $A$ [HH06].

With these definitions in place, it follows that Theorem 2.1.15 implies that singularities/pairs of strongly $F$-regular type (resp. of $F$-pure type) are KLT (resp. log canonical). We have the following converse statement.

**Theorem 2.1.16** ([Har98a, MS97, Har01, Smi97]). *A KLT pair has strongly $F$-regular type.*

[11] Meaning $R$ is a normal with $\Delta$ a $\mathbb{Q}$-divisor on $X = \operatorname{Spec} R$.
[12] That is, dense in the maximal spectrum of $A$.

The analogous statement implying that log canonical pairs have $F$-pure type is true subject to the Weak Ordinarity Conjecture [MS11, Tak13, BST13a].

## 2.2 Finite torsors

In one sentence, a torsor is a fppf $G$-bundle for a group-scheme $G$. It is convenient to begin by collecting the relevant notions about group-schemes and their action on schemes and rings. For the most part, we will follow the treatment in [Mil17], [Tat97] and [Mon93]. The proofs of all the statements, and definitions, can be found in these references.

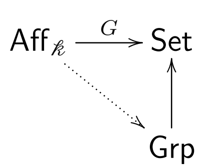
### 2.2.1 Affine group-schemes

Let $\mathscr{k}$ be a field, or more generally a ground ring. All fibered and tensor products are defined over $\mathscr{k}$ unless otherwise explicitly stated. Recall that to an affine $\mathscr{k}$-scheme $X$, we associate a covariant functor (its functor of points) $X : \mathsf{Alg}_{\mathscr{k}} \to \mathsf{Set}$, which is given by

$$X(-) = \mathrm{Hom}_{\mathsf{Alg}_{\mathscr{k}}}(\mathscr{O}(X), -),$$

where $\mathsf{Alg}_{\mathscr{k}}$ denotes the category of $\mathscr{k}$-algebras. In fact, this association $X \mapsto X(-)$ gives a fully faithful covariant functor from the category of affine $\mathscr{k}$-schemes $\mathsf{Aff}_{\mathscr{k}}$ to the dual category of $\mathsf{Alg}_{\mathscr{k}}$, say $\mathscr{Y} : \mathsf{Aff}_{\mathscr{k}} \to \mathsf{Alg}_{\mathscr{k}}^{\vee}$. This is just a restatement of Yoneda's lemma. In other words, the Yoneda's functor $\mathscr{Y}$ realizes $\mathsf{Aff}_{\mathscr{k}}$ as a full subcategory of $\mathsf{Alg}_{\mathscr{k}}^{\vee}$, say $\mathscr{Y}(\mathsf{Aff}_{\mathscr{k}})$, which is called the category of *representable functors*.

A functor $G$ in $\mathsf{Alg}_{\mathscr{k}}^{\vee}$ is said to be *formally a group* or a *formal group over* $\mathscr{k}$ if it factors through the forgetful functor $\mathsf{Grp} \to \mathsf{Set}$:




Notice that this means that there are natural transformations $\nabla : G \times G \to G$ (*group product*), $e : * \to G$ (*identity*)[13], and $\iota : G \to G$ (*inversion*), such that the following

[13]Here $*$ denotes the functor $\mathsf{Alg}_{\mathscr{k}} \to \mathsf{Set}$ sending $R$ to $\{\emptyset\}$; it is the final object of the category $\mathsf{Alg}_{\mathscr{k}}^{\vee}$.

diagrams of natural transformations are commutative

$$
\begin{array}{ccc}
G \times G \times G & \xrightarrow{\nabla \times \mathrm{id}} & G \times G \\
{\scriptstyle \mathrm{id} \times \nabla}\downarrow & & \downarrow{\scriptstyle \nabla} \\
G \times G & \xrightarrow{\nabla} & G
\end{array}
\qquad
\begin{array}{ccc}
* \times G & \xrightarrow{e \times \mathrm{id}} & G \times G \\
{\scriptstyle p_2}\downarrow{\scriptstyle \cong} & \swarrow{\scriptstyle \nabla} & \\
G & &
\end{array}
\qquad
\begin{array}{ccccc}
G \times G & & \xrightarrow{\mathrm{id} \times \iota,\ \iota \times \mathrm{id}} & & G \times G \\
{\scriptstyle \Delta}\uparrow & & & & \downarrow{\scriptstyle \nabla} \\
G & \longrightarrow & * & \xrightarrow{e} & G
\end{array}
$$

where $\Delta$ denotes the *diagonal natural transformation.*

An *affine group over $\mathscr{k}$*, say $G$, is defined to be a representable functor in $\mathsf{Alg}_{\mathscr{k}}^{\vee}$ that is formally a group over $\mathscr{k}$. One also says that an affine scheme $G$ is an *affine group-scheme over $\mathscr{k}$* (or group-scheme for short) if the functor $\mathscr{Y}(G) = G(-)$ is an affine group.

By Yoneda's lemma, the above definition is equivalent to saying that $G$ is equipped with morphisms of $\mathscr{k}$-schemes $\nabla$, $e$, and $\iota$ satisfying the same formal set of commutative diagrams we had above. Of course, affine groups are represented by group-schemes and vice versa. We use these two equivalent perspectives interchangeably, hoping the context makes clear which one is being used.

A *(homo)morphism of affine groups* $G \to G'$ is a natural transformation, which additionally satisfies that $G(R) \to G'(R)$ is a homomorphism of groups for all $\mathscr{k}$-algebra $R$. Equivalently, if $G$, $G'$ are group-schemes, then a morphism of group-schemes $\varphi : G \to G'$ is a morphism of $\mathscr{k}$-schemes such that the square

$$
\begin{array}{ccc}
G & \xrightarrow{\varphi} & G' \\
{\scriptstyle \nabla}\uparrow & & \uparrow{\scriptstyle \nabla'} \\
G \times G & \xrightarrow{\varphi \times \varphi} & G' \times G'
\end{array}
$$

is commutative. This provides us with the category of affine groups over $\mathscr{k}$.

The category of affine groups enjoys many similarities with the category of groups; many of the basic aspects of the theory of groups carry over to group-schemes. For instance, we can talk about the full subcategory of *abelian* affine groups, whose objects are the affine groups $G : \mathsf{Aff}_{\mathscr{k}} \to \mathsf{Set}$ that factor through the forgetful functor $\mathsf{Ab} \to \mathsf{Set}$. This category is in fact an abelian category. Likewise, we have existence of fibered products and general small (inverse) limits.[14] This allows us to define the *kernel* of any homomorphism of affine groups $\varphi : G \to G'$ to be the left-hand sided

[14] These are realized by the fibered products and limits of schemes.

homomorphism in the following fibered product

$$\begin{array}{ccc} G \times_{G'} * & \longrightarrow & * \\ {\scriptstyle \ker\varphi}\downarrow & & \downarrow{\scriptstyle e'} \\ G & \xrightarrow{\varphi} & G' \end{array}$$

It turns out that $\ker\varphi$ is a closed immersion of schemes. One can even make sense of $G/\ker\varphi$ (back to this shortly).

Given and affine group $G$, there is a canonical algebra $\mathcal{O}(G)$ representing $G$. Indeed, denote by $\mathbb{A}^1 : \mathsf{Alg}_{\mathcal{k}} \to \mathsf{Set}$ the forgetful functor, then as a set $\mathcal{O}(G) := \mathrm{Mor}(G, \mathbb{A}^1)$. An element $f \in \mathcal{O}(G)$ can be then thought of as a compatible family of functions $f_R : G(R) \to R$ indexed by $\mathcal{k}$-algebras $R$. The compatibility means that given a homomorphism $\varphi : R \to R'$, the diagram

$$\begin{array}{ccc} G(R) & \xrightarrow{f_R} & R \\ {\scriptstyle G(\varphi)}\downarrow & & \downarrow{\scriptstyle \varphi} \\ G(R') & \xrightarrow{f_{R'}} & R' \end{array}$$

commutes. The addition and multiplication are then defined pointwise. For example, $f+f'$ is defined by $(f+f')_R(g) = f_R(g) + f'_R(g)$ for all $\mathcal{k}$-algebras $R$ and all $g \in G(R)$. Thus, $\mathcal{O}(G)$ becomes a commutative ring, and the $\mathcal{k}$-algebra structure $\mathcal{k} \to \mathcal{O}(G)$ is defined by $c_R(g) = c$ for all $R$ and $c \in G(R)$.

Moreover, there is a natural transformation $G \to \mathrm{Hom}_{\mathsf{Alg}_{\mathcal{k}}}(\mathcal{O}(G), -)$ given by $g \mapsto (f \mapsto f_R(g))$ for all $R$, $g \in G(R)$ and $f \in \mathcal{O}(G)$. In fact, $G$ is representable if and only if this natural transformation is a natural equivalence.

The above construction of the canonical algebra $\mathcal{O}(G)$ did not depend on $G$ being an affine group. However, if $G$ happens to be an affine group, then $\mathcal{O}(G)$ is a *commutative Hopf $\mathcal{k}$-algebra*. Since this third perspective will be our working one, we will spend several pages summarizing the most basic concepts, and terminology, of this subject.

### Hopf algebras

Let us remember what the category of unitary associative $\mathcal{k}$-algebras is. An associative $\mathcal{k}$-algebra with unit is a triple $(A, \Delta, u)$, where $A$ is a $\mathcal{k}$-module, and $\Delta : A \otimes A \to A$, $u : \mathcal{k} \to A$ are $\mathcal{k}$-linear maps, called product and unit, respectively, such that the following diagrams are commutative:

$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{\Delta\otimes \mathrm{id}} & A\otimes A \\ {\scriptstyle \mathrm{id}\otimes\Delta}\downarrow & & \downarrow{\scriptstyle \Delta} \\ A\otimes A & \xrightarrow{\Delta} & A \end{array} \qquad \begin{array}{ccc} \Bbbk\otimes A & \xrightarrow{u\otimes \mathrm{id}} & A\otimes A \\ {\scriptstyle \cong}\downarrow & \swarrow{\scriptstyle \Delta} & \\ A & & \end{array} \qquad \begin{array}{ccc} A\otimes \Bbbk & \xrightarrow{\mathrm{id}\otimes u} & A\otimes A \\ {\scriptstyle \cong}\downarrow & \swarrow{\scriptstyle \Delta} & \\ A & & \end{array}$$

Let $\tau : A \otimes A \to A \otimes A$ be the isomorphism $a_1 \otimes a_2 \mapsto a_2 \otimes a_1$. The algebra is said to be commutative if additionally, the following triangle commutes

$$\begin{array}{ccc} A\otimes A & \xrightarrow{\tau} & A\otimes A \\ & {\scriptstyle \Delta}\searrow \quad \swarrow{\scriptstyle \Delta} & \\ & A & \end{array}$$

A $\Bbbk$-linear map $\varphi : A_1 \to A_2$ is a homomorphism of $\Bbbk$-algebras if the following diagram commutes

$$\begin{array}{ccccc} \Bbbk & \xrightarrow{u_2} & A_2 & \xleftarrow{\Delta_2} & A_2\otimes A_2 \\ {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle \varphi}\uparrow & & \uparrow{\scriptstyle \varphi\otimes\varphi} \\ \Bbbk & \xrightarrow{u_1} & A_1 & \xleftarrow{\Delta_1} & A_1\otimes A_1 \end{array}$$

In this setting, an *ideal* $I$ of $A$ is defined to be a $\Bbbk$-submodule such that $\Delta(A\otimes I) \subset A$ and $\Delta(I \otimes A) \subset A$. We have that ideals are exactly the kernels of homomorphisms.

Now, by reversing arrows, we get the dual notion of coassociative $\Bbbk$-coalgebra with counit. A $\Bbbk$-module $C$ is a *$\Bbbk$-coalgebra* if it is endowed with $\Bbbk$-linear maps $\nabla : C \to C \otimes C$, $e : C \to \Bbbk$, called *coproduct* and *counit*,[15] satisfying a set of commutative diagrams dual to the ones we had for algebras. Namely,

$$\begin{array}{ccc} C\otimes C\otimes C & \xleftarrow{\nabla\otimes \mathrm{id}} & C\otimes C \\ {\scriptstyle \mathrm{id}\otimes\nabla}\uparrow & & \uparrow{\scriptstyle \nabla} \\ C\otimes C & \xleftarrow{\nabla} & C \end{array} \qquad \begin{array}{ccc} \Bbbk\otimes C & \xleftarrow{e\otimes \mathrm{id}} & C\otimes C \\ {\scriptstyle \cong}\uparrow & \nearrow{\scriptstyle \nabla} & \\ C & & \end{array} \qquad \begin{array}{ccc} C\otimes \Bbbk & \xleftarrow{\mathrm{id}\otimes e} & C\otimes C \\ {\scriptstyle \cong}\uparrow & \nearrow{\scriptstyle \nabla} & \\ C & & \end{array}$$

The coalgebra is said to be *cocommutative*[16] if additionally, $\nabla = \tau \circ \nabla$.

A $\Bbbk$-linear morphism $\varphi : C_1 \to C_2$ is a homomorphism of $\Bbbk$-coalgebras if the following diagram commutes

$$\begin{array}{ccccc} \Bbbk & \xleftarrow{e_2} & C_2 & \xrightarrow{\nabla_2} & C_2\otimes C_2 \\ {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle \varphi}\uparrow & & \uparrow{\scriptstyle \varphi\otimes\varphi} \\ \Bbbk & \xleftarrow{e_1} & C_1 & \xrightarrow{\nabla_1} & C_1\otimes C_1 \end{array}$$

[15] However, we will sometimes refer to them as *group product* and *identity*, respectively.

[16] Or abelian.

A $\Bbbk$-submodule of $I$ of $C$ is a *coideal* if it satisfies $\nabla(I) \subset A \otimes I + I \otimes A$. Likewise, a quotient $C/I$ inherits the coalgebra structure if and only if $I$ is a coideal. Thus, coideals correspond exactly to kernels of homomorphisms.

*Remark* 2.2.1. If $(A, \Delta, u)$ and $(A', \Delta', u')$ are $\Bbbk$-algebras as above, then $A \otimes A'$ becomes a $\Bbbk$-algebra with multiplication and unit given respectively by the compositions:

$$(A \otimes A') \otimes (A \otimes A') \xrightarrow{\cong} (A \otimes A) \otimes (A' \otimes A') \xrightarrow{\Delta \otimes \Delta'} A \otimes A',$$
$$\Bbbk \xrightarrow{\cong} \Bbbk \otimes \Bbbk \xrightarrow{u \otimes u'} A \otimes A'.$$

Similarly, if $(C, \nabla, e)$ and $(C', \nabla', e')$ are $\Bbbk$-coalgebras, then $C \otimes C'$ can be given the structure of coalgebra by:

$$C \otimes C' \xrightarrow{\nabla \otimes \nabla'} (C \otimes C) \otimes (C' \otimes C') \xrightarrow{\cong} (C \otimes C') \otimes (C \otimes C'),$$
$$C \otimes C' \xrightarrow{e \otimes e'} \Bbbk \otimes \Bbbk \xrightarrow{\cong} \Bbbk.$$

With the above remark in mind, we have the following proposition [Mon93, §1.3].

**Proposition 2.2.2** ([Mon93])**.** *Let $B$ be a $\Bbbk$-module so that $(B, \Delta, u)$ is a $\Bbbk$-algebra and $(B, \nabla, e)$ a $\Bbbk$-coalgebra. Then the following two statements are equivalent:*

*(a)* $\nabla : B \to B \otimes B$ *and* $e : B \to \Bbbk$ *are $\Bbbk$-algebra homomorphisms.*

*(b)* $\Delta : B \otimes B \to B$ *and* $u : \Bbbk \to B$ *are $\Bbbk$-coalgebra homomorphisms.*

Therefore, we have the following definitions.

**Definition 2.2.3** (Bialgebras and Hopf algebras)**.** (a) Let $(B, \Delta, u, \nabla, e)$ be as in Proposition 2.2.2. We say it is a $\Bbbk$-*bialgebra* if either of the equivalent conditions in Proposition 2.2.2 holds. It is said to be commutative (resp. cocommutative) if the underlying algebra (resp. coalgebra) is so. We have a notion of *biideals*; these are the submodule being simultaneously ideals and coideals. Biideals have the expected properties, *e.g.* quotients of bialgebras by biideals inherit the bialgebra structure, and biideals correspond to kernels of bialgebras homomorphims. Any module-theoretic notion on the bialgebra is referred to as the underlying algebra.

(b) Let $(B, \Delta, u, \nabla, e)$ be a $\Bbbk$-bialgebra. An *antipodal map* (or just *antipode* for short) is a $\Bbbk$-algebra homomorphism $\iota : B \to B$ for which *both*[17] of the following diagrams are commutative

$$\begin{array}{ccccc} B \otimes B & & \xrightarrow{\mathrm{id}\times\iota,\, \iota\times\mathrm{id}} & & B \otimes B \\ \nabla \big\uparrow & & & & \big\downarrow \Delta \\ B & \xrightarrow{e} & \Bbbk & \xrightarrow{u} & B \end{array}$$

As a matter of fact, any bialgebra admits at most one antipode.

(c) A $\Bbbk$-bialgebra admitting an antipodal map is known as a *Hopf algebra over* $\Bbbk$ (or Hopf $k$-algebra). A homomorphism of Hopf $\Bbbk$-algebras is defined simply as a homomorphism of $\Bbbk$-bialgebras (as the antipodal maps structures are automatically respected, *i.e.* a homomorphism of bialgebras commutes with the antipodes).

(d) A biideal $I$ of a Hopf algebra $H$ is said to be a *Hopf ideal* if $\iota(I) \subset I$. Then $H/I$ inherits the Hopf algebra structure if and only if $I$ is a Hopf ideal. Hopf ideals are the kernels of homomorphism of Hopf algebras. An example of this is the *augmentation ideal* $I_H$ of a Hopf algebra $H$; this is the kernel of its counit $e : H \to \Bbbk$.

We have the following equivalence of categories [Mil17, Corollary 3.7]

**Proposition 2.2.4.** *The functor $G \rightsquigarrow \mathcal{O}(G)$ gives an anti-equivalence between the category of affine groups over $\Bbbk$ and the category of commutative finitely generated Hopf $\Bbbk$-algebras. The quasi-inverse is of course given by* Spec*.*

Let $(C, \nabla, e)$ be a coalgebra over $\Bbbk$, then by taking duals with respect to $\Bbbk$, one gets $(C^\vee, \nabla^\vee, e^\vee)$ an algebra, not necessarily commutative, over $\Bbbk$. More precisely, the product of the algebra is given by the composition

$$C^\vee \otimes C^\vee \to (C \otimes C)^\vee \xrightarrow{\nabla^\vee} C^\vee$$

rather than just $\nabla^\vee$. Nevertheless, recall that the arrow $C^\vee \otimes C^\vee \to (C \otimes C)^\vee$ is natural.

[17] Note that there are two morphism over the top arrow.

For the dual statement to hold, *i.e.* dualizing an algebra $(A, \Delta, u)$ to get a coalgebra, we require the natural map $A^\vee \otimes A^\vee \to (A \otimes A)^\vee$ to be an isomorphism. This is the case if $\Bbbk$ is a field and $A$ is finite dimensional. Assuming this, by using

$$A^\vee \xrightarrow{\Delta^\vee} (A \otimes A)^\vee \leftrightarrow A^\vee \otimes A^\vee$$

as coproduct, the triple $(A^\vee, \Delta^\vee, u^\vee)$ becomes a coalgebra over $\Bbbk$.

In this way, if $(B, \Delta, u, \nabla, e)$ is a bialgebra over $\Bbbk$, its dual $(B^\vee, \nabla^\vee, e^\vee, \Delta^\vee, u^\vee)$ is a bialgebra as well. Algebra and coalgebra structures are interchanged. Furthermore, observe that the notion of antipode is self-dual, whereby this discussion extends to Hopf algebras.

In summary, dualizing $H \rightsquigarrow H^\vee$ results in an auto anti-equivalence of the category of finite Hopf algebras over a field $\Bbbk$.

### Group-theoretic notions

The anti-equivalence $G \rightsquigarrow \mathcal{O}(G)$ is a very fruitful one. For example, let $\varphi : G \to G'$ be a homomorphism of group-schemes, which is the same as a homomorphism of Hopf algebras $\varphi^\# : \mathcal{O}(G') \to \mathcal{O}(G)$. Then, the kernel of $\varphi$ defined in the previous section corresponds to the morphism of Hopf algebras $\mathcal{O}(G) \twoheadrightarrow \mathcal{O}(G)/\varphi^\#(I_{G'})$, where $I_{G'}$ is the augmentation ideal of $\mathcal{O}(G')$. Moreover, we may now define the *quotient homomorphisms* $G \to G/\ker\varphi$. This corresponds by definition to the homomorphism of Hopf algebras $\varphi^\#\big(\mathcal{O}(G')\big) \subset \mathcal{O}(G)$.

*Remark* 2.2.5. Any extension of finite commutative Hopf algebras is faithfully flat (this is a far from trivial fact). In fact, a homomorphism of group-schemes is faithfully flat if and only if the corresponding morphism of Hopf algebras is injective. Analogously, a homomorphism of group-schemes is a closed immersion if and only if it is surjective at the level of Hopf algebras. As a caveat, if the homomorphism is a closed immersion, then certainly the kernel is the trivial group. The converse does not hold in general, however. Nevertheless, it does hold if the base ring $\Bbbk$ is a field.

Furtheremore, an *affine subgroup* of an affine group $G$ is by definition an affine group represented by one of the quotient Hopf algebras of $\mathcal{O}(G)$. The latter are in bijection with the Hopf ideals of $\mathcal{O}(G)$.

Coming back to our starting example, the Hopf algebra $\varphi^\#\big(\mathcal{O}(G')\big)$ defines the *image* of $\varphi$, say $\operatorname{im}\varphi$. In this manner, $\varphi$ factors as

$$G \to \operatorname{im}\varphi \to G'$$

where $G \to \operatorname{im}\varphi$ is faithfully flat and $\operatorname{im}\varphi \to G'$ is a closed immersion.

Let $G \to G''$ be a faithfully flat homomorphism with kernel $G' \to G$. Observe that if $G \to Q$ is a homomorphism such that $G' \to G \to Q$ factors through the trivial homomorphism $* \to Q$, then it factors uniquely through $G \to G''$. Inspired by this, one calls the homomorphism $G \to G''$ the *quotient* of $G$ by $G'$, sometimes simply denoted by $G/G'$.

In general, if $G' \to G$ is a subgroup, we say that $G \to G''$ is a *quotient* of $G' \to G$ if $G' \to G \to G''$ factors through $* \to G''$ and is initial with this property. Quotients are then uniquely determined up to unique isomorphism. On the existence of quotients, say $\Bbbk$ is a field. Then, among faithfully flat homomorphism $G \to G''$ such that $G' \to G \to G''$ factors through $* \to G''$, there is a universal one. However, $G'$ happens to be its kernel if and only if $G'$ is *normal*. This property may serve as a definition of normality. Otherwise, a subgroup or more generally a closed immersion $G' \to G$ is *normal* if $G'(R)$ is a normal subgroup of $G(R)$ for all $k$-algebra $R$. Normal subgroups are precisely the kernels of faithfully flat homomorphisms.

With this being said, a sequence

$$* \to G' \to G \to G'' \to *$$

is called *exact*, if $G \to G''$ is faithfully flat and $G' \to G$ is its kernel. We also refer to it as a *short exact sequence*. The sequence is said to *split* if the quotient morphism admits a section. Then $G$ is a semi-direct product $G' \rtimes G''$.

An affine group-scheme is said to be *finite* if the underlying scheme is finite over $\Bbbk$. In case $\Bbbk$ is a field, the *order* is defined to be $o(G) := \dim_{\Bbbk} \mathscr{O}(G)$.

For a short exact sequence, as before, of finite group-schemes,

$$* \to G' \to G \to G'' \to *$$

we have additivity for the order function, that is $o(G) = o(G') + o(G'')$.

### Scheme-theoretic notions

Let $G$ be a finite group-scheme over a field $\Bbbk$. Let $A$ be the largest étale $\Bbbk$-subalgebra of $\mathscr{O}(G)$. This in fact exists and is a Hopf subalgebra of $\mathscr{O}(G)$. We denote by $\pi_0(G)$

the group-scheme associated to $A$. The faithfully flat homomorphism $G \to \pi_0(G)$ corresponding to the inclusion $A \subset \mathcal{O}(G)$ coincides with the structure map to connected components of $G$. The kernel of this homomorphism is $G^\circ$, the connected component containing the identity. Summing up, we have a short exact sequence,

$$* \to G^\circ \to G \to \pi_0(G) \to *$$

This sequence is unique among short exact sequences realizing $G$ as an extension of an étale group-scheme by a connected group-scheme.

If $\mathscr{k}$ is perfect, then this sequence splits. Hence, it realizes $G$ as a semidirect product $G^\circ \rtimes \pi_0(G)$. In particular, scheme-theoretically, $G$ is the product over $\mathscr{k}$ of $G^\circ$ and $\pi_0(G)$. At the level of $\mathscr{k}$-algebras, we then have that $\mathcal{O}(G)$ is the tensor product of $\mathcal{O}(G^\circ)$ and an étale (finite) $\mathscr{k}$-algebra $A$.

Furthermore, the scheme-theoretic structure of a *connected* finite group-scheme over a perfect field of positive characteristic $p > 0$ is well understood:

$$\mathcal{O}(G^\circ) \cong \mathscr{k}[t_1, ..., t_n] \Big/ \left(t_1^{p^{e_1}}, ..., t_n^{p^{e_n}}\right) \text{ for some integers } e_1, ..., e_n \geq 1.$$

Consequently, $\mathcal{O}(G^\circ)$ is a Gorenstein $\mathscr{k}$-algebra. The étale $\mathscr{k}$-algebra $A$ is also Gorenstein, for it is isomorphic to the product of finitely many separable field extensions of $\mathscr{k}$. Thus, since $\mathcal{O}(G)$ is the tensor product of two Gorenstein finite $\mathscr{k}$-algebras, it is Gorenstein too.

A well-known theorem by P. Cartier establishes that if $\mathscr{k}$ is a field of characteristic zero, then $G$ is reduced and in fact smooth over $\mathscr{k}$ [Mil17, Theorem 3.23], *c.f.* [DG70, II,§6.1]. In particular, if $G/\mathscr{k}$ is finite, then it is étale.

### Examples

Given a finite abstract group $G$, there are at least two functorial ways to obtain a finite Hopf $\mathscr{k}$-algebra from it. These are Cartier dual[18] to each other though. The first way gives a commutative Hopf algebra, and so a group-scheme that is called the *constant group-scheme*. It is denoted by $G$ by abuse of notation.

Concretely, since $G$ is a set and $\mathscr{k}$ is a commutative $\mathscr{k}$-algebra, we have that the set $\mathrm{Hom}_{\mathsf{Set}}(G, \mathscr{k})$ is naturally a $\mathscr{k}$-algebra. Recall that $\mathrm{Hom}_{\mathsf{Set}}(-, \mathscr{k})$ is a fully faithful contravariant functor $\mathsf{Set} \to \mathsf{Alg}_k$. However, if $G$ is an abstract group, then

[18] The Cartier dual of an abelian finite group-scheme $G/\mathscr{k}$ is defined to be $G^\vee := \mathrm{Spec}\big(\mathcal{O}(G)^\vee\big)$.

$\mathrm{Hom}_{\mathsf{Set}}(G, \Bbbk)$ can be endowed with a Hopf algebra structure. Indeed, the coproduct is given by

$$\begin{aligned} \nabla : \mathrm{Hom}_{\mathsf{Set}}(G, \Bbbk) \to \mathrm{Hom}_{\mathsf{Set}}(G \times G, \Bbbk) &\xleftarrow{\cong} \mathrm{Hom}_{\mathsf{Set}}(G, \Bbbk) \otimes \mathrm{Hom}_{\mathsf{Set}}(G, \Bbbk) \\ \gamma \mapsto \big((g,h) \mapsto \gamma(gh)\big) \end{aligned}$$

It is in the isomorphism $\mathrm{Hom}_{\mathsf{Set}}(G \times G, k) \xleftarrow{\cong} \mathrm{Hom}_{\mathsf{Set}}(G, \Bbbk) \otimes \mathrm{Hom}_{\mathsf{Set}}(G, \Bbbk)$ that we require $G$ to be finite. The identity $e : \mathrm{Hom}_{\mathsf{set}}(G, \Bbbk) \to \Bbbk$ is the evaluation at the identity-of-$G$ map. The antipode is defined by the rule $\iota(\gamma)(g) := \gamma(g^{-1})$ for all $\gamma \in \mathrm{Hom}_{\mathsf{Set}}(G, k)$ and $g \in G$.

In summation, we have a fully faithful contravariant functor $\mathrm{Hom}_{\mathsf{Set}}(-, \Bbbk)$ from the category of finite groups to the category of Hopf algebras over $\Bbbk$. Equivalently, we have a fully faithful covariant functor from the category of finite groups to the category of finite group-schemes over $\Bbbk$. Moreover, if $\Bbbk$ is separably closed, then the essential image of this functor is the full subcategory of étale group-schemes over $\Bbbk$.

On the other hand, the dual construction works for any abstract group $G$, but in contrast, it gives a cocommutative Hopf algebra rather than a commutative one. Nonetheless, it is commutative if and only if the group $G$ is abelian. For this construction, the underlined algebra is $\Bbbk[G]$, the group $k$-algebra of $G$. The group-product, identity, and antipode are given respectively by the quite simple rules

$$\nabla(g) = g \otimes g, \quad e(g) = 1, \quad \iota(g) = g^{-1}, \quad \text{for all } g \in G.$$

When $G$ is abelian, we denote the corresponding group-scheme by $D(G)$, so that $D$ is an additive contravariant fully faithful functor from the category of abelian groups to the category of (abelian) group-schemes over $\Bbbk$. In fact, the affine group of $D(G)$ is the functor $R \rightsquigarrow \mathrm{Hom}_{\mathsf{Grp}}(G, R^{\times})$.

Since $D$ is additive, it is enough to understand what it does to $\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ to know what it does to any other abelian group. We have that $D(\mathbb{Z})$ is the so-called *multiplicative group* $\mathbb{G}_{\mathrm{m}}$. This corresponds to the affine group $R \rightsquigarrow R^{\times}$, and is represented by the Hopf algebra $\mathscr{O}(\mathbb{G}_{\mathrm{m}}) = \Bbbk\big[\zeta, \zeta^{-1}\big]$. On the other hand, $D(\mathbb{Z}/n\mathbb{Z})$ is the *group of $n$-th roots of unity* $\boldsymbol{\mu}_n$. When it is viewed as an affine group, it is the functor $R \rightsquigarrow \{r \in R^{\times} \mid r^n = 1\}$. Its Hopf algebra is $\mathscr{O}(\boldsymbol{\mu}_n) = k[\zeta] \,/\, \big(\zeta^n - 1\big)$.

By the exactness of $D$, we see that when it hits the short exact sequence of abelian groups

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

it produces the exact sequence of abelian group-schemes

$$* \to \boldsymbol{\mu}_n \to \mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}} \to * \tag{2.1}$$

At the level of Hopf algebras, the faithfully flat homomorphism $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}$ is the inclusion $\mathcal{k}\left[\zeta, \zeta^{-1}\right] \to \mathcal{k}\left[\zeta, \zeta^{-1}\right]$ raising both $\zeta$ and $\zeta^{-1}$ to the $n$-th power. The augmentation ideal of $\mathcal{k}\left[\zeta, \zeta^{-1}\right]$ is the ideal $(\zeta - 1)$, so that the kernel of this map is realized by the Hopf ideal $\left(\zeta^n - 1\right)$, *i.e.* the kernel is $\mathcal{k}\left[\zeta, \zeta^{-1}\right] / \left(\zeta^n - 1\right) = \mathcal{k}[\zeta] / \left(\zeta^n - 1\right)$.

We had that the image $D(\mathsf{Ab})$ is a full subcategory of the category of abelian group-schemes, and its essential image is called the category of *diagonalizable group schemes*. Thus, a group-scheme is said to be *diagonalizable* if it is isomorphic to $D(G)$ for some abelian group $G$. One defines $\mathbb{D}_n := D(\mathbb{Z}^{\oplus n})$, which is the group of invertible diagonal matrices.

There is a very precise way to describe the quasi-inverse of $D$. This is done via the *functor of characters* $X = \mathrm{Hom}(-, \mathbb{G}_{\mathrm{m}})$. More in detail, if $G$ is a group-scheme, a *character* of $G$ is by definition a homomorphism $\chi : G \to \mathbb{G}_{\mathrm{m}}$. Equivalently, it is a homomorphism of Hopf algebras $\mathcal{O}(\mathbb{G}_{\mathrm{m}}) \to \mathcal{O}(G)$. This at the same time is equivalent to give a unit $\gamma$ of $\mathcal{O}(G)$ such that $\nabla(\gamma) = \gamma \otimes \gamma$. These elements are also called the *group-like elements* of the Hopf algebra. From this latter interpretation, it follows that the set of characters is an abelian group.

The group-scheme $\mathbb{G}_{\mathrm{m}}$ has an additive counterpart, namely the *additive group* $\mathbb{G}_{\mathrm{a}}$. This is defined as the forgetful functor $\mathsf{Alg}_{\mathcal{k}} \to \mathsf{Ab}$ sending a $\mathcal{k}$-algebra $R$ to its underline additive group $(R, +)$. It is represented by the Hopf algebra $\mathcal{k}[\xi]$ with structural maps determined by

$$\nabla(\xi) = \xi \otimes 1 + 1 \otimes \xi, \quad e(\xi) = 0, \quad \iota(\xi) = -\xi.$$

If $\mathcal{k}$ is a field of characteristic $p > 0$, one has the $e$-th iterate of the relative Frobenius homomorphism on $\mathcal{O}(\mathbb{G}_{\mathrm{a}})$. This is the map $\mathcal{O}(\mathbb{G}_{\mathrm{a}}) \to \mathcal{O}(\mathbb{G}_{\mathrm{a}})$ given by $\xi \mapsto \xi^q$. It is a morphism of Hopf $\mathcal{k}$-algebras as well. It is injective, so faithfully flat. The augmentation ideal of $\mathcal{O}(\mathbb{G}_{\mathrm{a}})$ is $(\xi)$. Thereby, the kernel of $F^e \colon \mathbb{G}_{\mathrm{a}} \to \mathbb{G}_{\mathrm{a}}$ is given by $\mathcal{O}(\mathbb{G}_{\mathrm{a}})/(\xi^q)$. The kernel thus obtained is the, Cartier self-dual, *infinitesimal group-scheme* $\boldsymbol{\alpha}_q$. It fits into a short exact sequence,

$$* \to \boldsymbol{\alpha}_q \to \mathbb{G}_{\mathrm{a}} \xrightarrow{F^e} \mathbb{G}_{\mathrm{a}} \to * \tag{2.2}$$

As a functor, $\boldsymbol{\alpha}_{p^e}$ is described as $R \rightsquigarrow \{r \in R \mid r^q = 0\}$.

Likewise, there is a short exact sequence

$$* \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_\mathrm{a} \xrightarrow{F-\mathrm{id}} \mathbb{G}_\mathrm{a} \to * \tag{2.3}$$

More classically, we have $\mathrm{GL}_n$ the functor sending $R$ to $\mathrm{GL}_n(R)$, the multiplicative group of nonsingular $n \times n$ matrices over $R$. $\mathrm{GL}_n$ is an affine group represented by $\mathscr{O}(\mathrm{GL}_n) = \Bbbk[t_{ij}, t] \;/\; \big(\det(t_{ij}) \cdot t - 1\big)$. Relevant for our forthcoming discussion are the following subgroups of $\mathrm{GL}_n$. Let $\mathbb{T}_n \subset \mathrm{GL}_n$ be the subgroup of upper triangular matrices and let $\mathbb{U}_n \subset \mathbb{T}_n$ be the subgroup of upper triangular matrices with 1 along the diagonal. Actually, $\mathbb{U}_n$ is a normal subgroup of $\mathbb{T}_n$ whose quotient is $\mathbb{D}_n \subset \mathrm{GL}_n$, the subgroup of diagonal matrices. So we have a short exact sequence

$$* \to \mathbb{U}_n \to \mathbb{T}_n \to \mathbb{D}_n \to * \tag{2.4}$$

### Trigonalizable groups

Let us begin by recalling the concept of *coradical* of a coalgebra. A coalgebra is said to be *(co)simple* if it has no proper subcoalgebras. As a matter of fact, every simple subcoalgebra of any coalgebra is finite dimensional. The *coradical* of a coalgebra $C$ is defined as the sum of all simple subcoalgebras of $C$ and denoted by $C_0$.

The same construction applies to Hopf algebras, under the caveat that the coradical of a Hopf algebra is not necessarily a Hopf subalgebra. However, we always have inclusions

$$\Bbbk \cdot 1 = \Bbbk[\{1\}] \subset \Bbbk[X(H)] \subset H_0 \subset H \tag{2.5}$$

where $X(H)$ here represents the abelian group of group-like elements of $H$. Notice $\Bbbk[X(H)]$ coincides with the sum of all *one-dimensional* simple subcoalgebras of $H$, as any one-dimensional simple subcoalgebra is of the form $\Bbbk \cdot h$ for some $h \in X(H)$.

Now we can use (2.5) to define *unipotent*, *linearly reductive*, and *trigonalizable* group-schemes. In concrete, a group-scheme $G$ is said to be *unipotent* if $\Bbbk \cdot 1 = \mathscr{O}(G)_0$. On the opposite extreme, $G$ is said to be *linearly reductive* if $\mathscr{O}(G)_0 = \mathscr{O}(G)$. If $\Bbbk[X(G)] = \mathscr{O}(G)_0$, then $G$ is called *trigonalizable*. In all these three cases, we have that $\mathscr{O}(G)_0$ is a Hopf subalgebra.

Notice that a group-scheme $G$ is diagonalizable if and only if $\Bbbk[X(G)] = \mathscr{O}(G)$. Hence, diagonalizable group-schemes are linearly reductive. Both unipotent and diagonalizable group-schemes are trigonalizable.

A group-scheme that is geometrically diagonalizable is said to be of *multiplicative type*. That is, $G$ is of multiplicative type if $G_{\mathscr{k}^{\mathrm{sep}}}$ is diagonalizable. The groups of multiplicative type are exactly the abelian linearly reductive groups. Furthermore, Nagata's theorem explains linearly reductive groups in terms of their connected-étale decomposition. Namely, Nagata's theorem establishes that $G$ is linearly reductive if and only if $p \nmid o\big(\pi_0(G)\big)$ and $G^\circ$ is of multiplicative type. Thus, if $\mathscr{k}$ is separably closed, $G$ is linearly reductive if and only if $\pi_0(G)$ is a constant group-scheme whose order is not divisible by $p$ and $G^\circ = D(\Gamma)$ for some abelian group $\Gamma$ whose torsion is divisible by $p$. There are at least two more ways to characterize groups of multiplicative type:

$$\begin{aligned} G \text{ is abelian and } \operatorname{Hom}(G, \mathbb{G}_\mathrm{a}) = 0 &\iff G \text{ is of multiplicative type} \\ &\iff G \text{ is abelian and } \mathscr{O}(G)^\vee \text{ is étale} \end{aligned}$$

For unipotent group-schemes, we know the following properties. We have that subgroups, quotients, and extensions of unipotent group-schemes are unipotent. Unipotency and geometric unipotency are equivalent notions. We have that $\mathbb{U}_n$ and its subgroups are unipotent. Conversely, every unipotent group-scheme is isomorphic to a subgroup of $\mathbb{U}_n$. This last fact is crucial for us because $\mathbb{U}_n$ has a normal series (in fact central)

$$\mathbb{U}_n = U_n^{(0)} \supset U_n^{(1)} \supset \cdots \supset U_n^{\left(n(n-1)/2\right)} = *$$

whose intermediate quotients are canonically isomorphic to $\mathbb{G}_\mathrm{a}$. Hence, given a unipotent group-scheme, we can realize it as a subgroup of $\mathbb{U}_n$, then intersect it with the above normal series to get a normal series for it whose quotients are all subgroups of $\mathbb{G}_\mathrm{a}$. Roughly speaking, $\mathbb{G}_\mathrm{a}$ and its subgroups are the building blocks for unipotent groups, and among the most basic ones are $\boldsymbol{\alpha}_p$ and $\mathbb{Z}/p\mathbb{Z}$ (these are the finite simple building blocks).

Trigonalizable group-schemes are the type of group-schemes upon which we will concentrate our attention in Chapter 5. Intuitively, for us, groups of multiplicative type and unipotent groups are like water and oil, respectively. In this analogy, trigonalizable groups are a mix of these where oil always floats to the top and water remains on bottom. This order will be important for us. More precisely, a group $G$ is trigonalizable if and only if there exists a normal unipotent subgroup such that its quotient is diagonalizable. In other words, trigonalizable group-schemes are the extension of a diagonalizable group by a unipotent one. For instance, from (2.4), we see that $\mathbb{T}_n$ is trigonalizable. In fact, a group-scheme is trigonalizable if and only if it is isomorphic to a subgroup of $\mathbb{T}_n$.

We have that subgroups and quotients of trigonalizable groups are trigonalizable. However, trigonalizable group-schemes are not closed under extensions. Crucially for us, trigonalizable groups are closed under small limits. In particular, if $G_1 \to G_0$ and $G_2 \to G_0$ are homomorphisms of trigonalizable groups, then $G_1 \times_{G_0} G_2$ is trigonalizable.

If $G$ is geometrically trigonalizable, we have that it contains a unique normal unipotent subgroup $G_{\mathrm{u}}$ whose quotient is of multiplicative type. So a group-scheme is geometrically trigonalizable if and only if it is an extension of a group of multiplicative type by a unipotent group. The subgroup $G_{\mathrm{u}}$ is also characterized by being the largest unipotent subgroup of $G$, *i.e.* it is characterized by containing any other unipotent subgroup of $G$. The formation of $G_{\mathrm{u}}$ commutes with base extensions. Actually, if $G \subset \mathbb{T}_n$, then $G_{\mathrm{u}}$ coincides with $G \cap \mathbb{U}_n$.

In conclusion, a trigonalizable group $G$ comes equipped with a canonical exact sequence

$$* \to G_{\mathrm{u}} \to G \to G/G_{\mathrm{u}} \to *$$

This sequence splits if either $\Bbbk$ is algebraically closed, or if $\Bbbk$ is perfect and $G_{\mathrm{u}}$ is connected. Notice that all abelian groups are trigonalizable. In this case, one may swap the role of $G_{\mathrm{u}}$ and $G/G_{\mathrm{u}}$ in the above exact sequence.

Observe that all trigonalizable groups are solvable. An example when the converse is not true is the following. Say the characteristic of $\Bbbk$ is 2 and let $G$ be the subgroup of $\mathrm{SL}_2$ of $2 \times 2$ matrices $(a_{ij})$ such that $a_{ii}^2 = 1$ and $a_{12}^2 = 0 = a_{21}^2$. Then $G$ fits into the following short exact sequence

$$* \to \boldsymbol{\mu}_2 \to G \to \boldsymbol{\alpha}_2 \times \boldsymbol{\alpha}_2 \to *$$

where the closed immersion is given by $\zeta \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & \zeta \end{pmatrix}$, whereas the faithfully flat quotient is given by

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto (a_{11}a_{12}\,,\, a_{21}a_{22})$$

### 2.2.2 Actions, quotients, and torsors

Let $Y$ be a finite type $B$-scheme. A *(right) action* of an affine group-scheme $G/\Bbbk$ on $Y/B$ is a $B$-morphism $\alpha\colon Y \times G \to Y$ such that:

$$\begin{array}{ccc} Y \times G \times G & \xrightarrow{\mathrm{id}\times\nabla} & Y \times G \\ {\scriptstyle \alpha\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \alpha} \\ Y \times G & \xrightarrow{\alpha} & Y \end{array} \qquad \begin{array}{ccc} Y \times * & \xrightarrow{\mathrm{id}\times e} & Y \times G \\ {\scriptstyle \cong}\downarrow & \swarrow{\scriptstyle \alpha} & \\ Y & & \end{array}$$

are commutative diagrams.

A $B$-morphism $f\colon Y \to X$ is *$G$-invariant* if the action $\alpha\colon Y \times G \to Y$ is also an $X$-morphism, meaning that the following diagram commutes

$$\begin{array}{ccc} Y \times G & \xrightarrow{\alpha} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{f} & X \end{array}$$

Let $\mathscr{C}$ be a full subcategory of $B$-schemes, then a *quotient* in $\mathscr{C}$ of the action $\alpha\colon Y \times G \to Y$ is a $G$-invariant morphism $q\colon Y \to X$ in $\mathscr{C}$ factoring uniquely any other $G$-invariant morphism in $\mathscr{C}$. If it exists, it is unique up to unique isomorphism.

A *$G$-torsor* is a faithfully flat (and locally of finite-type) morphism $q\colon Y \to X$ together with an action $\alpha\colon Y \times G \to Y$ such that $q$ is $G$-invariant and moreover, the induced morphism $\alpha \times p\colon Y \times G \to Y \times_X Y$ is an isomorphism. In other words, a $G$-torsor is a $G$-bundle in the fppf-topology. Since $G/\Bbbk$ is affine, we have that isomorphism classes of $G$-torsors over $X$ are functorially classified by the pointed-set $\check{H}^1(X_{\mathrm{ft}}, G)$; see [Mil80, Chapter III, §4], [Gir71]. The distinguished point in $\check{H}^1(X_{\mathrm{ft}}, G)$ is given by the class of the trivial $G$-torsor $X \times G \to X$. If $\phi\colon G \to H$ is a homomorphism of group-schemes over $\Bbbk$, then one defines the map of pointed-sets $\check{H}^1(\phi)\colon \check{H}^1(X_{\mathrm{ft}}, G) \to \check{H}^1(X_{\mathrm{ft}}, H)$ as follows. If $Y \to X$ is a torsor, then $\check{H}^1(\phi)(Y \to X)$ is by definition *contracted product* $Y \wedge^G H_X$. Roughly speaking, $Y \wedge^G H_X$ is the quotient of $Y \times H$ by the diagonal right action of $G$, say $(y,h)\cdot g \mapsto \big(y\cdot g, \phi(g)^{-1}h\big)$, so

$$Y \wedge^G H := {Y \times H}\Big/{(yg,h) \sim \big(y, \phi(g)h\big)}$$

The right action of $H$ on $Y \wedge^G H$ is given by the rule $(y,h_1)\cdot h_2 = \big(y, h_2h_1\big)$. One verifies that $Y \wedge^G H \to X$, say $(y,h) \mapsto f(y)$, is an $H$-torsor under that action.

In case $G$ is abelian, $\check{H}^1(X_{\mathrm{ft}}, G)$ coincides with the derived-functor cohomology abelian group $H^1(X_{\mathrm{ft}}, G)$. Moreover, given a short exact sequence

$$* \to G \to G \to G'' \to *$$

we have an exact sequence of pointed sets

$$* \to G'(X) \to G(X) \to G''(X) \xrightarrow{\delta^0} \check{H}^1\big(X_{\mathrm{ft}}, G'\big) \to \check{H}^1\big(X_{\mathrm{ft}}, G\big) \to \check{H}^1\big(X_{\mathrm{ft}}, G''\big)$$

that can be continued using second nonabelian cohomology [Gir71, Deb77]

$$\cdots \to \check{H}^1(X_{\mathrm{ft}}, G'') \xrightarrow{\delta^1} \check{H}^2\big(X_{\mathrm{ft}}, G'\big) \to \check{H}^2\big(X_{\mathrm{ft}}, G\big) \to \check{H}^2\big(X_{\mathrm{ft}}, G''\big)$$

Nevertheless, in the abelian case, these coincide with the long exact sequence from derived-functor abelian cohomology with respect to the fppf site.

*Remark* 2.2.6 (On the Galois correspondence for torsors). Let $G \to G''$ be a faithfully flat homomorphisms of finite group-schemes over $\Bbbk$, and let $G' \to G$ be the corresponding kernel. Let $Y \to X$ be a $G$-torsor, then one has that the restricted action $Y \times G' \to Y$ realizes the morphisms $Y \to Y \wedge^G G''$, say $y \mapsto (y, e)$, as a $G'$-torsor.

Next, we would like to restate the axioms for actions in a way that will be useful for us later on. Notice we can base change all $(G, \nabla, e, \iota)$ by $Y/\Bbbk$ to get $(G_Y, \nabla_Y, e_Y, \iota_Y)$ a group-scheme over $Y$. Thus, the two axioms for right actions translate into the commutativity of

$$\begin{array}{ccc} G_Y \times_Y G_Y & \xrightarrow{\nabla_Y} & G_Y \\ {\scriptstyle \alpha \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \alpha} \\ G_Y & \xrightarrow{\alpha} & Y \end{array}$$

and to say $e_Y$ is a section of $\alpha$. In the ring-theoretic setting of Hopf algebras, if $Y = \operatorname{Spec} S$, then an action of $G$ on $Y$ is nothing but a *coaction* of $\mathscr{O}(G)$ on $S$, namely a homomorphism $\alpha^{\#} \colon S \to S \otimes \mathscr{O}(G)$ satisfying the following two commutative diagrams:

$$\begin{array}{ccc} S \otimes \mathscr{O}(G) & \xrightarrow{\mathrm{id} \otimes \nabla} & S \otimes \mathscr{O}(G) \otimes \mathscr{O}(G) \\ {\scriptstyle \alpha^{\#}}\uparrow & & \uparrow{\scriptstyle \alpha^{\#} \otimes \mathrm{id}} \\ S & \xrightarrow{\alpha^{\#}} & S \otimes \mathscr{O}(G) \end{array} \qquad \begin{array}{ccc} S \otimes \Bbbk & \xleftarrow{\mathrm{id} \otimes e} & S \otimes \mathscr{O}(G) \\ {\scriptstyle \cong}\uparrow & \nearrow{\scriptstyle \alpha^{\#}} & \\ S & & \end{array}$$

The *ring of coinvariants* $S^G$ is defined by $\{s \in S \mid \alpha^{\#}(s) = s \otimes 1\}$.

As before, by base changing the Hopf algebra $\mathscr{O}(G)$ by $S/\Bbbk$, we get the Hopf $S$-algebra $\mathscr{O}(G_S)$ associated to the group-scheme $G_S = S \times G$ over $S$. Explicitly, the coproduct $\nabla_S$ is given by composition of $\mathrm{id} \otimes \nabla$ with $S \otimes \mathscr{O}(G) \otimes \mathscr{O}(G) \xrightarrow{\cong} \big(S \otimes \mathscr{O}(G)\big) \otimes_S \big(S \otimes \mathscr{O}(G)\big)$, whereas the identity $e_S \colon \mathscr{O}(G_S) \to S$ is given by $\mathrm{id} \otimes e$. Thus, the coaction axioms can be written in a more compact and convenient fashion as:

$$\begin{CD} \mathscr{O}(G_S) @>\nabla_S>> \mathscr{O}(G_S)\otimes_S\mathscr{O}(G_S) \\ @A{\alpha^{\#}}AA @AA{\alpha^{\#}\otimes\mathrm{id}}A \\ S @>{\alpha^{\#}}>> \mathscr{O}(G_S) \end{CD} \qquad \begin{array}{ccc} S & \xleftarrow{e_S} & \mathscr{O}(G_S) \\ {\scriptstyle\mathrm{id}}\uparrow & \overset{\alpha^{\#}}{\nearrow} & \\ S & & \end{array}$$

It is important to notice that the second axiom implies that $\alpha^{\#}$ is injective.

Here are some key facts we will make use of throughout: If $\mathscr{C}$ is the category of affine $B$-schemes, then quotients exist and are given (locally of affine charts) by the spectra of the rings of coinvariants $S^G \subset S$. If additionally $G$ is finite, then quotients are finite and onto. In fact, $S^G \subset S$ is an integral extension; see [Mon93, Theorem 4.2.1] and the references there, and also see [Mum08, Chapter III, §12, Theorem 1]. Finally, notice torsors are always quotients of their respective actions, even in the category of all $B$-schemes. This follows from the fact that faithfully flat morphisms of finite type are strict epimorphisms, [Mil80, Chapter I, §2, Theorem 2.17].

*Remark* 2.2.7 (On actions and coactions). We mentioned above that an action of $G$ on $\operatorname{Spec} S$ is the same as a coaction of $\mathscr{O}(G)$ on $S$. This is also equivalent to an action of $\mathscr{O}(G)^{\vee}$ on $S$. For definitions and further details, see [Mon93, Chapter 4]. Roughly, an *action* of a Hopf algebra $H$ on $S$ is a $\Bbbk$-linear map $\beta \colon H \otimes S \to S$ (*i.e.* a left $H$-module structure on $S$) satisfying a pair of axioms dual to the ones we had for coactions. Namely, the diagram

$$\begin{array}{ccccccc} & & & H\otimes S & & & \\ & \overset{\mathrm{id}\otimes\Delta_S}{\nearrow} & & & & \overset{\beta}{\searrow} & \\ H\otimes S\otimes S & & & & & & S \\ & \underset{\nabla\otimes\mathrm{id}}{\searrow} & & & & \underset{\Delta_S}{\nearrow} & \\ & & H\otimes H\otimes S\otimes S & \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} H\otimes S\otimes H\otimes S & \xrightarrow{\beta\otimes\beta} & S\otimes S & \end{array}$$

commutes; where $\Delta_S$ is the diagonal homomorphism of $S$. The second axiom is requiring that $h \cdot 1 = e(h) \cdot 1 = e(h)$ for all $h \in H$. The *ring of invariants* is defined by $\{s \in S \mid h \cdot s = e(h)s \text{ for all } h \in H\}$.

If $H$ is finite dimensional, an action (resp. coaction) of $H$ is the same as a coaction (resp. action) of $H^\vee$, in such a way that rings of invariants and coinvariants are the same in either perspective. Indeed, if $H$ coacts on $S$ by $\alpha^\#\colon S \to S \otimes H$, then $H^\vee$ coacts on $S$ via the composition

$$H^\vee \otimes S \xrightarrow{\mathrm{id}\otimes\alpha} H^\vee \otimes S \otimes H \xrightarrow{\mathrm{id}\otimes\tau} H^\vee \otimes H \otimes S \xrightarrow{\langle -,-\rangle\otimes\mathrm{id}} S$$

More explicitly, if $\eta \in H^\vee$ and $s \in S$, then $\eta \cdot s = (\mathrm{id} \otimes \eta)\big(\alpha^\#(s)\big)$ for all $\eta \in H^\vee$, $s \in S$.

From this, it is clear that the coinvariant elements are invariant. For, if $\alpha^\#(s) = s \otimes 1$, then $\eta \cdot s = \eta(1)s$ for all $\eta$. The converse, however, is a bit more subtle, as it relies on $H$ being finite dimensional. Indeed, to check that two elements (*e.g.* $\alpha^\#(s)$ and $s \otimes 1$) in the *finite rank free* $S$-module $S \otimes H$ are the same, it suffices to show their images under $\mathrm{id} \otimes \eta$ are the same for all $\eta \in H^\vee$, for these maps generate the $S$-dual module of $S \otimes H$. The upshot is that invariants are coinvariants due to indirect reasons. Thus, it is in principle easier to show an element is an invariant than a coinvariant, although these are *a posteriori* equivalent.

Furthermore, given an action $H^\vee \otimes S \to S$, its associated coaction is only defined after choosing a $\mathscr{k}$-basis for $H$. Indeed, if $h_1, ..., h_d$ form a basis with corresponding dual basis $\eta_1, ..., \eta_d$, then the coaction is given by the rule $s \mapsto (\eta_1 \cdot s) \otimes h_1 + \cdots + (\eta_d \cdot s) \otimes h_d$.

## 2.3 Finite torsors over singularities

Let $(R, \mathfrak{m}, \mathscr{k}, K)$ be a strictly local[19] $\mathscr{k}$-rational normal[20] domain of dimension at least 2 over an algebraically closed field $\mathscr{k}$. For short, we refer to any such a $(R, \mathfrak{m}, \mathscr{k}, \mathscr{l}, K)$ as a $\mathscr{k}$-rational germ. Let $X = \operatorname{Spec} R$ and fix $Z \subset X$ a closed subscheme of codimension at least 2, with open complement $U \subset X$ and defining ideal $I$. In this dissertation, we are interested in understanding to what extent there are finite group-schemes $G$ over $\mathscr{k}$ such that the restriction map of $G$-torsors

$$\varrho^1_X(G)\colon \check{H}^1(X_{\mathrm{ft}}, G) \to \check{H}^1(U_{\mathrm{ft}}, G)$$

is not surjective. The goal of this section is to describe how our problem can be reduced to a ring-theoretic setting. Namely, we want to explain why if $\varrho^1_X(G)$ is not

[19]That is, strictly Henselian, which means Henselian with $\mathscr{k}$ separably closed.
[20]However, for the present discussion, we only required $R$ to be an $\mathbf{S}_2$ local domain.

surjective, then there exists a local finite extension $(R, \mathfrak{m}, \Bbbk, K) \subset (S, \mathfrak{n}, \Bbbk)$ with $S$ a $\mathbf{S}_2$ local ring, and a faithfully flat homomorphism $G \twoheadrightarrow G'$ with $(G')^\circ = G^\circ$, such that: $G'$ acts on $S$ in such a way that $R = S^{G'}$, and $\operatorname{Spec} S \to X$ induces a $G'$-torsor over $U$, but not everywhere. That is, the pullback of $\operatorname{Spec} S \to X$ to $U$ does not belong to the image of $\varrho^1_X(G')$.

This is basically done by taking integral closures. To this end, we begin with the following simple observations. Let $h\colon V \to U$ be a finite morphism with $U \subset X = \operatorname{Spec} R$; by its *integral closure* we mean the finite morphism $\widetilde{h}\colon Y \to X$ where $Y = \operatorname{Spec} S$ and $S := H^0(\mathscr{O}_V, V)$. Note that taking integral closures is functorial on finite $U$-schemes. Also observe that the pullback of $\widetilde{h}$ to $U$ recovers $h$. On the other hand, if $h : V \to U$ does come from restricting a $G$-torsor $Y \to X$, then it happens to be the integral closure of $h$, thus the following lemma is in order.

**Lemma 2.3.1** (Extending actions across integral closure)**.** *Let $h\colon V \to U$ be a $G$-torsor with action $\alpha\colon V \times G \to V$. Then $\alpha$ extends across the integral closure to a unique action $\widetilde{\alpha}\colon Y \times G \to X$ such that $\widetilde{h}\colon Y \to X$ is its quotient morphism. Moreover, $H^0(Y, \mathscr{O}_Y)$ is an $\mathbf{S}_2$ semi-local ring. We say that $h$* extends across the integral closure *if $\widetilde{h}$ is a torsor.*

*Proof.* We need the coaction of $\alpha$ on global sections to give $\widetilde{\alpha}$. For this, notice $R = H^0(U, \mathscr{O}_U)$, when $R$ is an $\mathbf{S}_2$ ring of dimension at least 2. Note $S = H^0(V, \mathscr{O}_V)$ is $\mathbf{S}_2$ as well. Indeed, since $h\colon V \to U$ is a faithfully-flat finite morphism, $h_*\mathscr{O}_V$ is an $\mathbf{S}_2$ $\mathscr{O}_U$-module. Then, $\iota_* h_* \mathscr{O}_V$ is an $\mathbf{S}_2$ $\mathscr{O}_X$-module by [Har94, Theorem 1.12], but $H^0(X, \iota_* h_* \mathscr{O}_V) = S$ by definition. Since $S$ is an $\mathbf{S}_2$ $R$-module, it is an $\mathbf{S}_2$ ring, for restriction of scalars under finite maps does not change depth.

Thus, $S \otimes \mathscr{O}(G)$ is also $\mathbf{S}_2$, for it is a finite free extension of $S$. Since $V \times G \subset Y \times G$ is an open subscheme whose complement has codimension at least 2, it follows $H^0(V \times G, \mathscr{O}_{V \times G}) = S \otimes \mathscr{O}(G)$. Thus, the coaction of $\alpha$ on global sections induces a coaction $\alpha^{\#}(V)\colon S \to S \otimes \mathscr{O}(G)$, which gives the desired action. ☕

*Remark* 2.3.2. Let $R = S^G \subset S$ be a $G$-quotient. As in Corollary 5.1.11, let $W$ be the open subscheme of $\operatorname{Spec} R$ over which $Y = \operatorname{Spec} S$ is a torsor and suppose $W$ contains no codimension-1 points. Since we have that $R$ is $\mathbf{S}_2$, it follows $R = H^0(W, \mathscr{O}_W)$. However, we do not necessarily have that $H^0(Y_W, \mathscr{O}_{Y_W})$ is $S$ unless, for example, $S$ was $\mathbf{S}_2$ to start with.

The previous lemma makes precise that our problem reduces to study the question to what extent $G$-torsors $h\colon V \to U$ extend across the integral closure. However, $H^0(Y, \mathcal{O}_Y)$ might not be local. Nonetheless, in the following lemma, we show that we may restrict our attention to $V$ connected and so to $S$ local as $R$ is Henselian.

**Lemma 2.3.3.** *Let $h\colon V \to U$ be a finite $G$-torsor as before, it is dominated by a finite $G'$-torsor $h'\colon V' \to U$ where $V'$ is connected. More precisely, there is an equivariant finite morphism $V' \to V$ factoring $h'$ through $h$.*[21] *Moreover, $(G')^\circ \cong G^\circ$.*

*Proof.* Consider the connected-étale canonical decomposition $G = G^\circ \rtimes \pi_0(G)$. Then we have that the image of $h\colon V \to U$ under $\check{H}^1(U_{\mathrm{ft}}, G) \to \check{H}^1\big(U_{\mathrm{ft}}, \pi_0(G)\big)$ gives a $\pi_0(G)$-torsor $W \to U$, and a $G^\circ$-torsor $V \to W$ factoring $h\colon V \to W \to U$; see Remark 2.2.6. Since $W \to U$ is étale, it is dominated by any of its connected components; let $W' \subset W$ be one of them. By further domination, we may take $W' \to U$ to be generically Galois. We choose $V \times_W W'$ to be our $V'$.

Now, by [Nor82, Chapter II, Lemma 1] and [EV10, Proposition 2.2], we have that $V'$ is a $G \times_{\pi_0(G)} \mathrm{Gal}(W'/U) =: G'$ torsor over $U$, for $X$ is integral and is endowed with a $\Bbbk$-rational point.[22] Notice that the connected component at the identity remains unchanged, for scheme-theoretically $G = G^\circ \times \pi_0(G)$. It only remains to explain why $V'$ is connected. For this, notice $V' \to W'$ is a $G^\circ$-torsor and consider the following claim.

**Claim 2.3.4.** *Torsors over connected schemes for connected group-schemes are connected.*

*Proof of claim.* Since $\Bbbk = \Bbbk^{\mathrm{sep}}$, connectedness is the same as geometric connectedness. Set $X_2 \to X_1$ a finite $G_0$-torsor with both $X_1$ and $G_0$ connected. Since $G_0$ is geometrically connected, the number of connected components of $X_2 \times G_0$ is the same as the number of connected components of $X_2$; see for example [Sta18, Tag 0385]. On the other hand, the number of connected components of $X_2 \times_{X_1} X_2$ should be at least the square of the number of connected components of $X_2$, for $X_1$ is connected. Then $X_2$ has only one connected component. This proves the claim. ☕

[21] However, we do not mean $V' \to V$ is surjective nor dominant.

[22] In [Nor82] M. Nori proved that if $V_i$ are finite $G_i$-torsors over $U$, $i = 0, 1, 2$, and $f_i\colon V_i \to V_0$, $i = 1, 2$, are equivariant maps, then $V_1 \times_{V_0} V_2$ is a $G_1 \times_{G_0} G_2$-torsor over $U$ provided that $U$ is integral and $U(\Bbbk) \neq \emptyset$. In our case, $U$ is integral but $U(\Bbbk) = \emptyset$. This is remedied in [EV92] by using that $U \subset X$, and $X(\Bbbk) \neq \emptyset$.

This proves the lemma. ☕

Observe that in Lemma 2.3.3, if $h' \colon V' \to U$ extends across integral closure, so does $h \colon V \to U$; see [EV10, Proposition 2.3] for further details. In summary, we have proved the following.

**Proposition 2.3.5** (Reduction to local algebra)**.** *Let $h \colon V \to U$ be a $G$-torsor that is not the restriction of a $G$-torsor $Y \to X$. Then there exists* $\operatorname{Spec}(S, \mathfrak{n}, \mathscr{k}) \to X$ *a local $G'$-quotient that is a torsor over $U$ but not everywhere. Moreover, $G$ and $G'$ have isomorphic connected components at the identity.*

*Remark* 2.3.6. It is worth mentioning that the residue fields stay the same because $\mathscr{k}$ was assumed algebraically closed; this is deliberately done to ensure that every cover is endowed with a $\mathscr{k}$-rational point lying over $x$. We want the existence of these covers to depend on the geometry of the singularity and not on arithmetic issues coming from the groundfield.

### 2.3.1 The local étale fundamental group

When we restrict our attention to étale group-schemes over $\mathscr{k}$, our problem simplifies considerably. First of all, if $G$ is étale over $\mathscr{k}$, then $\check{H}^1(X_{\mathrm{ft}}, G)$ is a singleton, *i.e.* every $G$-torsor over $X$ is trivial. Then, our problem amounts to study when $\check{H}^1(U_{\mathrm{ft}}, G)$ is trivial. *i.e.* a singleton. However, we have functorial bijections of pointed sets

$$\operatorname{Hom}_{\mathrm{cont}}\big(\pi_1^{\text{ét}}(U), G\big) \Big/ G \xrightarrow{\cong} \check{H}^1(U_{\mathrm{ft}}, G)$$

where $G$ acts on $\operatorname{Hom}_{\mathrm{cont}}\big(\pi_1^{\text{ét}}(U), G\big)$ by conjugation[23]; for further details, see [GW10, Section 11.5], [Mil80, Chapter I, Remark 5.4 and Chapter III, Corollary 4.7, Remark 4.8]. The marked point of the set on the left is the trivial homomorphism of groups.

Here $\pi_1^{\text{ét}}(U)$ is the étale fundamental group of $U$ with base point $\operatorname{Spec} K^{\mathrm{sep}} \to U$, as defined in [Gro63, Exposé V], or well [Mur67]. It is a profinite topological group characterized by the above natural bijections. More generally, it is the fundamental group classifying or pro-representing the Galois category of finite étale cover over $U$. By our preliminary discussion on integral closures, or simply by Zariski's Main Theorem [Mil80, Chapter I, Theorem 1.8], we see that the Galois category of finite

[23] Thinking now of $G$ as an abstract finite group.

étale covers over $U$ is equivalent to the Galois category of finite covers over $X$, which are étale over $U$, *c.f.* [Art77].

As customary for normal schemes, by choosing the base point to be a separable closure of the fraction field, we may compute or express the étale fundamental group as the inverse limit $\varprojlim \mathrm{Gal}(L/K)$ over all finite Galois extensions $K \subset L \subset K^{\mathrm{sep}}$ such that the integral closure of $U$ inside $L$ is étale [Mil80, Example 5.2 (b)]. Equivalently, the limit traverses all finite Galois extensions such that the integral closure of $R$ in $L$ is étale over $U$.

We will refer to $\pi_1^{\text{ét}}(U)$ as the *local étale fundamental group* of $X$. It measures and classifies isomorphism classes of generically Galois covers over $X$ that are étale over $U$. It is worth noting that we do not need $\Bbbk$ to be perfect nor $\mathfrak{m}$ to be $\Bbbk$-rational to define and study $\pi_1^{\text{ét}}(U)$.

As a historical aside, it is worth saying that the study of local fundamental groups has a long history. It has early origins in the study of resolution of singularities based on the work of S. Abhyankar. It also dates back to the celebrated theorem by D. Mumford [Mum61] on the equivalence between smoothness and triviality of the local fundamental group, over the complex numbers. Mumford's result was generalized to the algebraic setting by H. Flenner [Fle75], *c.f.* [CS93, Corollary 5].

# Chapter 3

# Transformation rule for the $F$-signature

In this chapter, we introduce a transformation rule for the $F$-signature under finite covers. This transformation rule is largely responsible for the main results of this dissertation. We dedicate to it an entire chapter due to its central role in this work. We also consider this transformation rule interesting in its own right. As an example of this, in Chapter 6, we give separate applications of this transformation rule.

**Theorem 3.0.1** (Transformation rule for the $F$-signature under finite morphisms)**.** *Let $(R, \mathfrak{m}, \mathscr{k}, K) \subset (S, \mathfrak{n}, \mathscr{l}, L)$ be a local extension of normal domains with corresponding morphism of schemes $f\colon Y \to X$. Let $\Delta$ be an effective $\mathbb{Q}$-divisor on $X$. Suppose that there is a nonzero morphism of $S$-modules $\tau\colon S \to \operatorname{Hom}_R(f_*S, R) = \omega_{S/R}$ such that $T \coloneqq \tau(1)$ is surjective, $T(\mathfrak{n}) \subset \mathfrak{m}$, and $\Delta^* \coloneqq f^*\Delta - D_T$ is effective on $Y$. Then the following equality holds*

$$[\mathscr{l} : \mathscr{k}] \cdot s(S, \Delta^*) = [L : K] \cdot s(R, \Delta).$$

*Furthermore, if $(R, \mathfrak{m}, \mathscr{k}, K) \subset (S, \mathfrak{n}, \mathscr{l})$ is just a local extension, with $R$ just a domain and $S$ a reflexive $R$-module,*[1] *and $\tau$ is an isomorphism, then*

$$[\mathscr{l} : \mathscr{k}] \cdot s(S) = \dim_K S_K \cdot s(R)$$

*where $K \to S_K$ is the generic fiber of $R \subset S$.*

*Remark* 3.0.2. With the same setup as in Theorem 3.0.1, notice that $\tau$ is an isomorphism if and only if $D_T$ is the zero divisor. In particular, the extension $R \subset S$ is quasi-étale[2] if and only the $S$-linear map $S \to \omega_{S/R}$ sending $1 \in S$ to the trace map $\operatorname{Tr}_{S/R}$ is an isomorphism.

[1] For example, if $R$ is an $\mathbf{S}_2$ domain and $S$ an $\mathbf{S}_2$ local ring.

[2] Meaning étale in codimension 1.

*Remark* 3.0.3. Observe that with the same setup as in Theorem 3.0.1, if $\tau$ is an isomorphism, then the residual degree $[\ell : \Bbbk]$ is nothing but the free rank of the reflexive $R$-module $S$. Indeed,[3]

$$\mathrm{frk}_R f_*S = \lambda_R\big(\mathrm{Hom}_R(f_*S,R) \,/\, \mathrm{Hom}_R(f_*S,\mathfrak{m})\big) = \lambda_R\big(S\cdot T/\mathfrak{n}\cdot T\big) = \lambda_R(\ell) = [\ell : \Bbbk]$$

where $\mathrm{Hom}_R(f_*S,\mathfrak{m}) = \mathfrak{n}\cdot T$ precisely because $T$ is surjective and $T(\mathfrak{n}) \subset \mathfrak{m}$. In fact, $\mathrm{Hom}_R(f_*S,\mathfrak{m}) \subset \mathfrak{n}\cdot T$ because $T(S) = R$,[4] and conversely, $\mathrm{Hom}_R(f_*S,\mathfrak{m}) \supset \mathfrak{n}\cdot T$ because $T(\mathfrak{n}) \subset \mathfrak{m}$.

As an immediate consequence, we obtain the following result.

**Corollary 3.0.4** (Perseverance of strong $F$-regularity)**.** *Suppose we are in the same setting as Theorem 3.0.1. Then the pair $(S, f^*\Delta)$ is strongly $F$-regular if and only if the pair $(R,\Delta)$ is strongly $F$-regular. In particular, if $R$ is strongly $F$-regular, then $S$ is a normal domain.*

*Remark* 3.0.5. The "only if" direction in Corollary 3.0.4 was known before for general split extensions; see for example [SS10, Proposition 6.3]. To the best of the author's knowledge, only partial results were known for the converse of [SS10, Proposition 6.3]; see [HiWiY02, Proposition 1.4].

Before proceeding with the proof, we make some general comments on the nature of the hypothesis that will be used on the proof.

First of all, we recall Grothendieck duality for finite morphisms [Har66, Chapter III, §6], [Har77, Chapter III, Exercise 6.10]. Let $R \to S$ be a module-finite homomorphism of rings with $f\colon Y \to X$ the corresponding finite morphism of schemes. One defines the functor $f^!\colon R\text{-mod} \to S\text{-mod}$ by the rule $f^!M \coloneqq \mathrm{Hom}_R(f_*S, M)$. Also, one defines a natural transformation $\mathrm{Tr}\colon f_*f^! \to \mathrm{id}$ given by $\mathrm{Tr}_M\colon f_*f^!M \to M$, where $\mathrm{Tr}_M(\rho) \coloneqq \rho(1)$ for all $\rho \in \mathrm{Hom}_R(f_*S, M)$. Then one has the following fundamental result.

**Theorem 3.0.6** (Grothendieck duality for finite morphisms)**.** *Let $R \to S$ be a module-finite homomorphisms of rings with $f\colon Y \to X$ the corresponding finite morphism of schemes. Then the morphism of $R$-modules*

$$\xi = \xi(M,N)\colon f_*\,\mathrm{Hom}_S\big(N, f^!M\big) \to \mathrm{Hom}_R\big(f_*N, M\big), \quad \psi \mapsto \mathrm{Tr}_M \circ f_*\psi$$

[3] Using [BST12, Proposition 3.5, Lemma 3.6].

[4] Indeed, if $T(s\cdot -)\colon f_*S \to R$ is not surjective, then $s$ cannot be a unit, because otherwise, $T(sS) = T(S) = R$.

*is a natural isomorphism on both* $M \in R\text{-mod}$ *and* $N \in S\text{-mod}$*. In fact, its inverse* $\zeta = \zeta(M,N)$ *is given by* $\zeta(\vartheta)(n) := \vartheta(- \cdot n)$ *for all* $n \in N$ *and* $\vartheta \in \operatorname{Hom}_R(N,M)$*. In other words,*

$$\big(\zeta(\vartheta)(n)\big)(s) \coloneqq \vartheta(s \cdot n)$$

*for all* $s \in S$*,* $n \in N$*, and* $\vartheta \in \operatorname{Hom}_R(N,M)$*.*

We apply Grothendieck duality for finite morphims in the following way. Let $M = R$ and $N = F^e_* f^! R = F^e_* \omega_{S/R}$, then we get an isomorphism of $R$-modules

$$\begin{aligned} f_* \operatorname{Hom}_S\left(F^e_*\omega_{S/R}, \omega_{S/R}\right) &\to \operatorname{Hom}_R\left(f_* F^e_* \omega_{S/R}, R\right) = \operatorname{Hom}_R\left(F^e_* f_* \omega_{S/R}, R\right) \\ \psi &\mapsto \operatorname{Tr}_R \circ f_* \psi \end{aligned}$$

In particular, for all $\varphi \in \mathscr{C}^R_e$, there exists a unique $\varphi^! \in \operatorname{Hom}_S\left(F^e_*\omega_{S/R}, \omega_{S/R}\right)$ making the following diagram of $R$-modules commutative

$$\begin{array}{ccc} F^e_* f_* \omega_{S/R} & \xrightarrow{f_*\varphi^!} & f_*\omega_{S/R} \\ {\scriptstyle F^e_* \operatorname{Tr}_R}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{Tr}_R} \\ F^e_* R & \xrightarrow{\varphi} & R \end{array}$$

Indeed, $\varphi^! = \zeta\big(\varphi \circ F^e_* \operatorname{Tr}_R\big)$, or more concisely

$$\begin{aligned} \varphi^!(F^e_*\rho)(s) = \big(\varphi \circ F^e_* \operatorname{Tr}_R\big)(s \cdot F^e_*\rho) = \big(\varphi \circ F^e_* \operatorname{Tr}_R\big)\big(F^e_*(s^q \cdot \rho)\big) &= \varphi\big(F^e_* \operatorname{Tr}_R(s^q \cdot \rho)\big) \\ &= \varphi\big(F^e_* \rho(s^q)\big) \end{aligned}$$

for all $s \in S$ and $\rho \in \omega_{S/R} = \operatorname{Hom}_R(f_*S, R)$.

Now, recall that if we have a nonzero section $\tau \colon S \to \omega_{S/R}$, say $T = \tau(1)$, then it extends to an isomorphism $\tau \colon S(D_T) \to \omega_{S/R}$, $\tau(l) = T(l \cdot -)$. In fact, $S(D_T) \subset L$ is the largest $S$-submodule of $L$ that is mapped into $R$ under $T_K \colon L \to K$, where $T_K = T \otimes_R K$. In other words, $T \colon f_*S \to R$ can be extended to a map $T \colon f_*S(D_T) \to R$. Furthermore, we have the following commutative triangle of $R$-modules

$$\begin{array}{ccc} f_*S(D_T) & \xrightarrow{f_*\tau} & f_*\omega_{S/R} \\ & {\scriptstyle T}\searrow \quad \swarrow{\scriptstyle \operatorname{Tr}_R} & \\ & R & \end{array}$$

expressing that $T$ and $\mathrm{Tr}_R$ are identified under the isomorphism $\tau\colon S(D_T) \to \omega_{S/R}$. Consequently, we get natural isomorphisms of $R$-modules

$$\xi'_N\colon f_* \operatorname{Hom}_S\big(F^e_* N, S(D_T)\big) \to \operatorname{Hom}_R\big(F^e_* f_* N, R\big), \quad \psi \mapsto T \circ f_*\psi \tag{3.1}$$

Let $\varphi \in \mathscr{C}^R_e$. Then the composition $\varphi \circ F^e_* T\colon F^e_* f_* S \to F^e_* R \to R$ corresponds to the divisor $D_T + f^* D_\varphi$ on $Y$. Hence, it is the restriction of a (unique) map in

$$\operatorname{Hom}_R\big(F^e_* f_* S(D_T + f^* D_\varphi), R\big).$$

In fact, it is the composition

$$F^e_* f_* S(D_T + f^* D_\varphi) \xrightarrow{F^e_* T} F^e_* R(D_\varphi) \xrightarrow{\varphi} R.$$

In this way, by applying (3.1) with $N = S(D_T + f^* D_\varphi)$, we have that there exists a unique $\psi$ making the following diagram commutative

$$\begin{array}{ccc} F^e_* f_* S(D_T + f^* D_\varphi) & \xrightarrow{f_*\psi} & f_* S(D_T) \\ {\scriptstyle F^e_* T}\downarrow & & \downarrow{\scriptstyle T} \\ F^e_* R(D_\varphi) & \xrightarrow{\varphi} & R \end{array} \tag{3.2}$$

Nonetheless, we can untwist the top morphism to get a commutative square

$$\begin{array}{ccc} F^e_* f_* S(D_T + f^* D_\varphi - qD_T) & \xrightarrow{f_*\psi} & f_* S \\ {\scriptstyle F^e_* T}\downarrow & & \downarrow{\scriptstyle T} \\ F^e_* R(D_\varphi) & \xrightarrow{\varphi} & R \end{array}$$

Therefore, if the divisor $D_T + f^* D_\varphi - qD_T = f^* D_\varphi - (q-1)D_T$ is effective, *i.e.* if $f^*\Delta_\varphi - D_T$ is effective, we can restrict to $S$ to get a commutative diagram

$$\begin{array}{ccc} F^e_* f_* S & \xrightarrow{f_*\psi} & f_* S \\ {\scriptstyle F^e_* T}\downarrow & & \downarrow{\scriptstyle T} \\ F^e_* R & \xrightarrow{\varphi} & R \end{array} \tag{3.3}$$

with $D_\psi = f^* D_\varphi - (q-1)D_T$, equivalently $\Delta_\psi = f^*\Delta_\varphi - D_T$. Moreover, $\psi$ is unique under these conditions.

Following K. Schwede and K. Tucker [ST14, §5], we say that $\varphi$ *has a transpose along* $T$, or that $\psi$ *is a transpose of* $\varphi$ *along* $T$.

Conversely, say that we have a commutative diagram (3.3). Then there is a relation

$$D_T + f^*D_\varphi = D_\psi + (F^e)^*D_T = D_\psi + qD_T$$

in other words,

$$f^*D_\varphi - (q-1)D_T = D_\psi \geq 0$$

So, we have that $f^*\Delta_\varphi - D_T$ is effective and equal to $\Delta_\psi$. In particular,

$$\begin{aligned}\psi \in \operatorname{Hom}_S\big(F^e_*S(D_\psi), S\big) &= \operatorname{Hom}_S\big(F^e_*S(D_T + f^*D_\varphi - qD_T), S\big)\\ &\cong \operatorname{Hom}_S\big(F^e_*S(D_T + f^*D_\varphi), S(D_T)\big)\end{aligned}$$

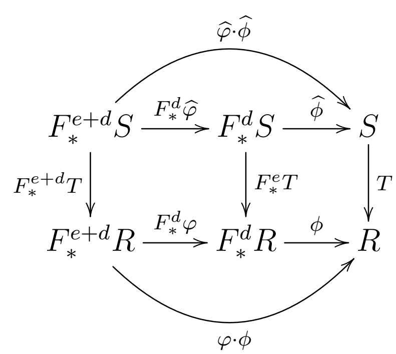
in such a way that when $\psi$ is realized as a map in $\operatorname{Hom}_S\big(F^e_*S(D_T + f^*D_\varphi), S(D_T)\big)$, it is the unique map making (3.2) commutative.

Summing up, we have recovered the Transposition criterion [ST14, Theorem 5.7].

**Proposition 3.0.7** (Schwede–Tucker Transposition Criterion)**.** *Let $(R, \mathfrak{m}, \mathscr{k}, K) \subset (S, \mathfrak{n}, \mathscr{l}, L)$ be a local extension of normal domains with corresponding morphism of schemes $f\colon Y \to X$. Let $\tau\colon S \to \omega_{S/R}$, say $\tau(1) = T$, be a nonzero section. Then $\varphi \in \mathscr{C}^R_e$ has a transposition along $T$ if and only if $\Delta^*_\varphi \coloneqq f^*\Delta_\varphi - D_T$ is effective. In that case, the transpose is unique, say $\widehat{\varphi}$, and $\Delta_{\widehat{\varphi}} = \Delta^*_\varphi$.*

*Furthermore, if $\Delta$ is an effective $\mathbb{Q}$-divisor on $X$ such that $\Delta^* \coloneqq f^*\Delta - D_T \geq 0$, then we get in this way a mapping $\varphi \mapsto \widehat{\varphi}$ from $\mathscr{C}^\Delta$ to $\mathscr{C}^{\Delta^*}$ which is a homomorphism of Cartier $R$-algebras.*

*Proof.* It only remains to prove that $\varphi \mapsto \widehat{\varphi}$ is a homomorphism of Cartier algebras. The $R$-bilinearity of this mapping is clear. To see that $\widehat{\varphi} \cdot \widehat{\phi} = \widehat{\varphi \cdot \phi}$ for $\varphi \in \mathscr{C}^R_e$ and $\phi \in \mathscr{C}^R_d$, consider the diagram




Then note that since the inner two squares are commutative, then so is the outer rectangle.

*Remark* 3.0.8 (Connection with Blickle–Stäbler's pullback functor). Let $f^*\mathscr{C}_e^{\Delta}$ be the (right) $S$-span of the image of $\mathscr{C}^{\Delta}$ in $\mathscr{C}^{\Delta^*}$ under the homomorphic mapping $\varphi \mapsto \widehat{\varphi}$. One can prove that $f^*\mathscr{C}_e^{\Delta}$ is a Cartier $S$-algebra that realizes the pullback operation defined by M. Blickle and A. Stäbler in [BS16]. We will not need this in what follows, however.

So far, we have only studied the hypothesis that we are given a nonzero section $\tau \colon S \to \omega_{S/R}$. Let us discuss now the significance of the other two hypothesis. For this, let us consider the isomorphism (3.1) with

$$\begin{aligned} N = S\big(D_T + \lceil (q-1)f^*\Delta \rceil\big) &= S\big(D_T + \lceil (q-1)(\Delta^* + D_T)\rceil\big) \\ &= S\big(\lceil (q-1)\Delta^*\rceil + qD_T\big) \\ &= S\big(\lceil (q-1)f^*\Delta\rceil + D_T\big) \end{aligned}$$

together with the projection formula isomorphism

$$\operatorname{Hom}_S\Big(F^e_* S\big(\lceil (q-1)\Delta^*\rceil + qD_T\big), S(D_T)\Big) \cong \operatorname{Hom}_S\Big(F^e_* S\big(\lceil (q-1)\Delta^*\rceil\big), S\Big) = \mathscr{C}_e^{\Delta^*}$$

to get an isomorphism

$$\xi' \colon f_*\mathscr{C}_e^{\Delta^*} \to \operatorname{Hom}_R\Big(F^e_* f_* S\big(\lceil (q-1)f^*\Delta\rceil + D_T\big), R\Big), \quad \psi \mapsto T \circ f_*\psi \tag{3.4}$$

where we realize

$$\operatorname{Hom}_R\Big(F^e_* f_* S\big(\lceil (q-1)f^*\Delta\rceil + D_T\big), R\Big) \subset \operatorname{Hom}_R(F^e_* f_* S, R)$$

via restriction $S \subset S\big(\lceil (q-1)f^*\Delta\rceil + D_T\big)$. In this way, we consider,

$$\begin{aligned} &\operatorname{Hom}_R\Big(F^e_* f_* S\big(\lceil (q-1)f^*\Delta\rceil + D_T\big), R\Big)^{\mathrm{ns}} \\ &:= \operatorname{Hom}_R\Big(F^e_* f_* S\big(\lceil (q-1)f^*\Delta\rceil + D_T\big), R\Big) \cap \operatorname{Hom}_R(f_* S, \mathfrak{m}). \end{aligned}$$

Now, observe that if $T(\mathfrak{n}) \subset \mathfrak{m}$, then

$$\xi'\left(f_*\Big(\mathscr{C}_e^{\Delta^*}\Big)^{\mathrm{ns}}\right) \subset \operatorname{Hom}_R\Big(F^e_* f_* S\big(\lceil (q-1)f^*\Delta\rceil + D_T\big), R\Big)^{\mathrm{ns}}.$$

Conversely, notice that if $T$ is surjective, then $\xi'(\psi) = T \circ \psi$ is surjective if $\psi$ is also. In other words,

$$\xi'\left(f_*\Big(\mathscr{C}_e^{\Delta^*}\Big)^{\mathrm{ns}}\right) \supset \operatorname{Hom}_R\Big(F^e_* f_* S\big(\lceil (q-1)f^*\Delta\rceil + D_T\big), R\Big)^{\mathrm{ns}}.$$

Putting these two observations together, we have the equality

$$\xi'\Big(f_*\big(\mathscr{C}_e^{\Delta^*}\big)^{\mathrm{ns}}\Big) = \Big(\xi'\big(f_*\mathscr{C}_e^{\Delta^*}\big)\Big)^{\mathrm{ns}} \tag{3.5}$$

which simply means that $\psi \in f_*\mathscr{C}_e^{\Delta^*}$ is surjective if and only if $\xi'(\psi)$ is surjective in (3.4).[5]

This in particular applies to $\psi = \widehat{\varphi}$ for $\varphi \in \mathscr{C}_e^{\Delta}$. Hence, $\widehat{\varphi}$ is surjective if and only if $\xi'(\widehat{\varphi}) = T \circ \widehat{\varphi} = \varphi \circ F_*^e T$ is surjective. However, the surjectivity of this latter map is equivalent to the surjectivity of $\varphi$ for $T$ is surjective. Hence, $\widehat{\varphi}$ is surjective if and only if $\varphi$ is surjective. Therefore, we get the "if" direction of the following result analogous to Corollary 3.0.4.

**Scholium 3.0.9** (Perseverance of $F$-purity)**.** *With the same setting as in Theorem 3.0.1, the pair $(S, \Delta^*)$ is $F$-split if the pair $(R, \Delta)$ is $F$-split. The converse holds in case $(q-1)\Delta$ is integral for $q-1$ sufficiently divisible. Otherwise, let $D$ be a divisor on $Y$ as in Lemma 3.0.10 below. Then the converse also holds if there exists a nonzero $s \in S(-D)$ not in the splitting prime of $(S, \Delta^*)$.*

*Proof.* We only need to explain the "only if" direction, for the converse direction follows from the fact that if $\varphi \in \mathscr{C}_e^{\Delta}$ is an $F$-splitting, then so is its transpose $\widehat{\varphi}$ along $T$, as we discussed in the previous paragraph.

In case $\Delta = 0$, this is well known; see for example [SZ15, Proposition 1.10]. Indeed, if $R \to S$ is just a split homomorphism, then if $F_*^e S \to S$ is a splitting, we can compose it with any splitting map $T' \colon S \to R$ to get a splitting map $F_*^e S \to R$ that when composed with $F_*^e R \to F_*^e S$ gives a splitting $F_*^e R \to R$.

In our case, we need to argue why this strategy works in the divisor setting as well, under either of the two scenarios proposed in the statement of the scholium. Notice, however, the second one is just a generalization of the first one, for if $(q-1)\Delta$ is integral for $q-1$ sufficiently divisible, then we can take $D = 0$. Then, the condition we can find $0 \neq s \in S(-D) = S$ not in the splitting prime of $(S, \Delta^*)$ is just a restatement of the $F$-purity of the pair $(S, \Delta^*)$.

Since $T \colon f_*S \to R$ is surjective, we know there is $s' \in S$ such that $T(s') = 1$. Next, let $\psi : F_*^e S\big(\lceil (q-1)\Delta^* \rceil\big) \to S$ be so that $\psi(F_*^e s) = s'$, which exists because

[5] Notice that the surjectivity of $\psi$ and $\xi'(\psi)$ is considered with respect to the common domain $F_*^e S$. Nonetheless, the same argument can be used to show the equivalence between the surjectivity of $\psi$ and $\xi'(\psi)$ with respect to the domains $F_*^e S\big(\lceil (q-1)\Delta^* \rceil\big)$ and $F_*^e S\big(\lceil (q-1) f^*\Delta + D_T \rceil\big)$, respectively.

$(S, \Delta^*)$ is $F$-pure and $s$ is not in the splitting prime of $(S, \Delta^*)$. Then the composition $T \circ (\psi \cdot s)$ when restricted to $F^e_* R$ gives a splitting $F^e_* R \to R$. It remains to explain why it belongs to $\mathscr{C}^{\Delta}_e$. For this, it suffices to show $rs \in S\big(\lceil (q-1)\Delta^* \rceil + qD_T\big)$ if $r \in R\big(\lceil (q-1)\Delta \rceil\big)$. Indeed, we know that the $R$-liner map $T \circ \psi \colon F^e_* f_* S \to R$ can be extended to

$$F^e_* f_* S\big(D_{T\circ\psi}\big) = F^e_* f_* S\big(D_\psi + qD_T\big) \supset F^e_* f_* S\big(\lceil (q-1)\Delta^* + D_T \rceil\big).$$

Now, the fact $s \cdot R\big(\lceil (q-1)\Delta \rceil\big) \subset S\big(\lceil (q-1)\Delta^* \rceil + qD_T\big)$ follows from the following divisorial estimates

$$\begin{aligned} \operatorname{div}_S rs = f^* \operatorname{div}_R r + \operatorname{div}_S s \geq -f^* \lceil (q-1)\Delta \rceil + D &\geq -\lceil (q-1) f^*\Delta \rceil \\ &= -\lceil (q-1)\Delta^* \rceil - (q-1)D_T \\ &\geq -\lceil (q-1)\Delta^* \rceil - qD_T. \end{aligned}$$

This proves the result. ☕

The following is a technical lemma bounding the error made in commuting pullbacks and roundups.

**Lemma 3.0.10.** *Let $f \colon Y \to X$ be a finite cover of normal schemes and $\Delta$ a $\mathbb{Q}$-divisor on $X$. Then there exists an integral divisor $D$ on $Y$ (depending only on the support of $\Delta$) such that*

$$0 \leq f^*\lceil \Delta' \rceil - \lceil f^*\Delta' \rceil \leq D$$

*for all $\mathbb{Q}$-divisor $\Delta'$ with the same support of $\Delta$. In fact, if $f \colon Y \to X$ is generically separable, then we may take $D$ to be the ramification divisor, which is further independent of $\Delta$.*

*Proof.* This is implicitly explained in [ST14, §2.2]. Indeed, the upshot of [ST14, §2.2, (2.1.2)] is the following: let $\Delta_{\text{red}} = \sum_i D_i$ with $D_i$ different prime divisors, then $D$ may be taken equal to $\sum e_j C_j$ where the sum traverses all the primes divisor on $Y$ whose image under $f$ is one of the $D_i$ supporting $\Delta$ and $e_j$ is the ramification index of $C_j$ along $f$.[6] If $f \colon Y \to X$ is separable, then this $D$ is bounded by Ram. ☕

We are now ready to prove the main result of this chapter.

[6]That is, $e_j$ is the ramification index of the extension of DVR's $\mathscr{O}_{X,f(C_j)} \subset \mathscr{O}_{Y,C_i}$. More precisely, $e_j$ is the order of a uniformizer for $\mathscr{O}_{X,f(C_j)}$ in $\mathscr{O}_{Y,C_j}$.

*Proof of Theorem 3.0.1.* Let $\delta = \dim R + \left[\mathscr{k}^{1/p} : \mathscr{k}\right] = \dim S + \left[\ell^{1/p} : \ell\right]$. Recall that we think of $\mathscr{C}_e^{\Delta}$ as the $p^{-e}$-linear maps $F_*^e R \to R$ admitting an extension to $F_*^e R\big(\lceil (q-1)\Delta \rceil\big) \to R$,[7] and similarly for $\mathscr{C}_e^{\Delta^*}$.

Since we are not assuming $L/K$ is separable, we will need to tweak $\mathscr{C}_e^{\Delta^*}$ a bit to compute the $F$-signature of the pair $(S, \Delta^*)$. We are going to utilize instead $\mathscr{D}_e \subset \mathscr{C}_e^{\Delta^*}$ the $p^{-e}$-linear maps $F_*^e S \to S$ that extend to a map $F_*^e S\big(\lceil (q-1)\Delta^* \rceil + D\big) \to S$, where $D$ is a fixed effective divisor as in Lemma 3.0.10. This modification is asymptotically insignificant and so it does not affect the $F$-signature; see [BST12, Lemma 4.17], [CST16, Lemma 2.7]. Indeed, if $c$ is any nonzero element of $S(-D)$, we have

$$\mathscr{C}_e^{\Delta^*} \cdot c \subset \mathscr{D}_e \subset \mathscr{C}_e^{\Delta^*}$$

for all $e > 0$.

Then, from our preliminary discussion, we have that

$$\begin{aligned} [\ell : \mathscr{k}] \cdot a_e(S, \mathscr{D}) = [\ell : \mathscr{k}] \cdot \lambda_S\big(\mathscr{D}_e \,/\, \mathscr{D}_e^{\mathrm{ns}}\big) &= \lambda_R\big(f_*\mathscr{D}_e \,/\, f_*\mathscr{D}_e^{\mathrm{ns}}\big) \\ &= \lambda_R\big(f_*\mathscr{D}_e \,/\, f_*\mathscr{D}_e^{\mathrm{ns}}\big) \\ &= \lambda_R\Big(\xi'\big(f_*\mathscr{D}_e\big) \Big/ \xi'\big(f_*\mathscr{D}_e^{\mathrm{ns}}\big)\Big) \\ &= \lambda_R\Big(\xi'\big(f_*\mathscr{D}_e\big) \Big/ \big(\xi'(f_*\mathscr{D}_e)\big)^{\mathrm{ns}}\Big) \end{aligned} \tag{3.6}$$

where $\xi'$ is as in (3.4) and

$$\xi'(f_*\mathscr{D}_e) = \mathrm{Hom}_R\Big(F_*^e f_* S\big(\lceil (q-1) f^*\Delta \rceil + D_T + D\big), R\Big)$$

the last step in (3.6) is just (3.5).

Let $\mathscr{E}_e \subset \mathrm{Hom}_R(F_*^e S, R)$ be the submodule consisting of maps $\vartheta \colon F_*^e S \to R$ such that for all $s \in S$, the map $F_*^e r \mapsto \vartheta(F_*^e rs)$ belongs to $\mathscr{C}_e^{\Delta}$. Then we claim

**Claim 3.0.11.** *There exists* $0 \neq c \in R$ *such that*

$$\mathscr{E}_e \cdot c \subset \xi'\big(f_*\mathscr{D}_e\big) \subset \mathscr{E}_e$$

*for all* $e > 0$.

*Proof.* For the second containment, we must show that $rs \in S\big(\lceil (q-1) f^*\Delta \rceil + D_T + D\big)$ for all $s \in S$ and $r \in R\big(\lceil (q-1)\Delta \rceil\big)$. Equivalently, we have to prove the inclusion

[7] Equivalently, maps $\varphi \colon F_*^e R \to R$ such that $D_\varphi \geq (q-1)\Delta$.

$R\big(\lceil (q-1)\Delta\rceil\big) \subset S\big(\lceil (q-1)f^*\Delta\rceil + D_T + D\big)$. However, this is clear for

$$\begin{aligned}\operatorname{div}_S(r) = f^*\operatorname{div}_R(r) \geq -f^*\lceil (q-1)\Delta\rceil &\overset{(\star)}{\geq} -\lceil (q-1)f^*\Delta\rceil - D\\ &\geq -\lceil (q-1)f^*\Delta\rceil - D_T - D.\end{aligned}$$

It is precisely because of $(\star)$ that we needed to tweak by $D$.[8] Hence, $\vartheta \in \mathscr{E}_e$.

It remains to construct $c$ making the first inclusion possible. For this, choose a free submodule $G = R^{\oplus[L:K]} \subset S$ and $0 \neq c_0 \in R$ such that $c_0 S \subset G$. It follows that

$$\begin{aligned}c_0 S\big(f^*\lceil (q-1)\Delta\rceil\big) = \Big(c_0 S \otimes_R R\big(\lceil (q-1)\Delta\rceil\big)\Big)^{\vee\vee} &\subset \Big(G \otimes_R R\big(\lceil (q-1)\Delta\rceil\big)\Big)^{\vee\vee}\\ &= \Big(R\big(\lceil (q-1)\Delta\rceil\big)\Big)^{\oplus[L:K]}\end{aligned}$$

where $(-)^{\vee\vee}$ denotes reflexification as an $R$-module (or equivalently, since it can be viewed as $\mathbf{S}_2$-ification, as an $S$-module where appropriate). Note that the equalities and containments can be checked in codimension 1 where they are obvious. This implies that if $\vartheta \in \mathscr{E}_e$, then $\vartheta \cdot c_0 \colon F^e_* S \to R$ can be extended to a map $F^e_* S\big(f^*\lceil (q-1)\Delta\rceil\big) \to R$.

Next, let $0 \neq c_1, c_2 \in R$ such that $\operatorname{div}_S c_1 \geq D_T$ and $\operatorname{div}_S c_2 \geq D$. Hence,

$$c_1 c_2 S\big(\lceil (q-1)f^*\Delta\rceil + D_T + D\big) \subset S\big(f^*\lceil (q-1)\Delta\rceil\big)$$

for if $s \in S\big(\lceil (q-1)f^*\Delta\rceil + D_T + D\big)$, then

$$\operatorname{div}_S(c_1c_2s) \geq D_T + D - \lceil (q-1)f^*\Delta\rceil - D_T - D \geq -f^*\lceil (q-1)\Delta\rceil.$$

Thus, $\vartheta\cdot(c_0c_1c_2)$ can be further extended to a map $F^e_* f_* S\big(\lceil (q-1)f^*\Delta\rceil + D_T + D\big) \to R$. In other words, letting $c \coloneqq c_0c_1c_2$, we have that $\vartheta \cdot c \in \xi'\big(f_*\mathscr{D}_e\big)$, as required. ☕

Therefore, by combining (3.6) and Claim 3.0.11, we have that it suffices to prove that the limit of

$$\frac{\lambda_R\big(\mathscr{E}_e \,/\, \mathscr{E}_e^{\mathrm{ns}}\big)}{q^\delta}$$

as $e$ goes to infinity is $[L:K]\cdot s(R,\Delta)$. This presumes a generalization of [Tuc12, Theorem 4.11].

[8] Although, if $L/K$ is separable and $T = T_{R/S}$, we might take $D = \mathrm{Ram} = D_{T_{S/R}}$ to control the error made in commuting pullbacks and roundups. Then the tweak would be unnecessary.

For this, Let $g := [L : K]$ and consider a short exact sequence

$$0 \to R^{\oplus g} \to S \to O \to 0$$

where $O$ is a torsion $R$-module, *i.e.* $\operatorname{Ann}_R O \neq 0$. Applying the exact functor $F^e_*$ followed by the left exact functor $\operatorname{Hom}_R(-, R)$, we get an exact sequence

$$0 \to \operatorname{Hom}_R(F^e_* S, R) \xrightarrow{\iota} \operatorname{Hom}_R\left(F^e_* R^{\oplus g}, R\right)$$

since $\operatorname{Hom}_R(F^e_* O, R) = 0$ (for $R$ is a domain and $O$, therefore $F^e_* O$, is torsion).

To avoid cumbersome notation in what follows, let us think of the injective map $R^{\oplus g} \to S$ as an actual inclusion $R^{\oplus g} \subset S$, so that $\iota$ is nothing but the restriction map. Thus, all we are pointing out is that this restriction map is injective, *i.e.* a map $\vartheta\colon F^e_* S \to R$ gets determined by its values at $F^e_* R^{\oplus g} \subset F^e_* S$. As before, by abuse of notation and for the sake of clarity, we think of $\iota$ as an actual inclusion as well. That is, we realize $\operatorname{Hom}_R(F^e_* S, R)$ inside $\operatorname{Hom}_R\left(F^e_* R^{\oplus g}, R\right)$ as the maps $\vartheta\colon F^e_* R^{\oplus g} \to R$ admitting a (necessarily unique) extension to a map $F^e_* S \to R$.

On the other hand, for any nonzero $c \in \operatorname{Ann}_R O$, we have $c \cdot S \subset R^{\oplus g}$. Therefore, following the aforementioned conventions, we obtain inclusions

$$\operatorname{Hom}_R\left(F^e_* R^{\oplus g}, R\right) \cdot c \subset \operatorname{Hom}_R(F^e_* S, R) \subset \operatorname{Hom}_R\left(F^e_* R^{\oplus g}, R\right) \tag{3.7}$$

for all $e > 0$, as any map $\vartheta\colon F^e_* R^{\oplus g} \to R$ when pre-multiplied (*i.e.* scaled on the right) by $c$ extends to $F^e_* S$. In other words, $\vartheta(F^e_* c \cdot -)$ can now be also evaluated at elements of $S$ and not just at elements of $R^{\oplus g} \subset S$.

Observe now that $\operatorname{Hom}_R\left(F^e_* R^{\oplus g}, R\right) \cong \operatorname{Hom}_R(F^e_* R, R)^{\oplus g} \supset \left(\mathscr{C}^{\Delta}_e\right)^{\oplus g}$. Moreover, under this identification, (3.7) restricts to

$$\left(\mathscr{C}^{\Delta}_e\right)^{\oplus g} \cdot c \subset \mathscr{E}_e \subset \left(\mathscr{C}^{\Delta}_e\right)^{\oplus g}$$

for all $e > 0$. On the other hand

$$\left(\left(\mathscr{C}^{\Delta}_e\right)^{\oplus g}\right)^{\mathrm{ns}} = \left(\left(\mathscr{C}^{\Delta}_e\right)^{\mathrm{ns}}\right)^{\oplus g}.$$

Therefore,

$$\begin{aligned}\lim_{e\to\infty}\frac{\lambda_R\big(\mathscr{E}_e \,/\, \mathscr{E}_e^{\mathrm{ns}}\big)}{q^\delta} &= \lim_{e\to\infty}\frac{\lambda_R\Bigg(\Big(\mathscr{C}_e^\Delta\Big)^{\oplus g} \Bigg/ \Bigg(\Big(\mathscr{C}_e^\Delta\Big)^{\mathrm{ns}}\Bigg)^{\oplus g}\Bigg)}{q^\delta}\\ &= \lim_{e\to\infty}\frac{\lambda_R\Bigg(\Big(\mathscr{C}_e^\Delta \Big/ \big(\mathscr{C}_e^\Delta\big)^{\mathrm{ns}}\Big)^{\oplus g}\Bigg)}{q^\delta}\\ &= \lim_{e\to\infty}\frac{g\cdot\lambda_R\Big(\mathscr{C}_e^\Delta \Big/ \big(\mathscr{C}_e^\Delta\big)^{\mathrm{ns}}\Big)}{q^\delta}\\ &= g\cdot s(R,\Delta)\end{aligned}$$

as desired. This proves the transformation rule. ☕

**Example 3.0.12** (Rational double points)**.** In the paper [Art77], M. Artin proved that in characteristic $p \geq 7$, rational double points[9] are all quasi-étale quotients of the smooth germ by the action of a finite subgroup $G \subset \mathrm{SL}_2$. More precisely, he proved that in those characteristics, rational double points are equisingular reductions mod $p$ of the complex rational double points.[10] In particular, Theorem 3.0.1 applies with $T = \mathrm{Tr}_{S/R}$, the Reynold's operator, and $\Delta = 0$. Thus, for a classic rational point, we have that the $F$-signature is $1/\#G$. Concretely, $s(A_n) = 1/(n+1)$ for $n \geq 1$, $s(D_n) = 1/4(n-2)$ for $n \geq 4$, $s(E_6) = 1/24$, $s(E_7) = 1/48$, and $s(E_8) = 1/120$. Compare with [HL02, Example 18] and [Hun13, Example 6.11] where the $F$-signature of these singularities is computed by substantially different methods.

**Example 3.0.13** (Veronese subrings)**.** Let $S = \bigoplus_{e\in\mathbb{N}} S_e$ be an $\mathbb{N}$-graded $\mathscr{k}$-algebra. Let $R = S^{(d)} \subset S$ be its $d$-th Veronese subring, *i.e.* $R = S^{(d)} = \bigoplus_{e\in\mathbb{N}} S_{de}$. We see directly that $S = S^{(d)} \oplus \bigoplus_{d\nmid e} S_e$. That is, the inclusion $R \subset S$ splits as $R$-modules. Let $T\colon S \to R$ be the splitting map (so surjective). In fact, $\omega_{S/R} = \mathrm{Hom}_R(S,R)$ is freely generated by $T$. It is not very hard to see Theorem 3.0.1 works in the graded setting as well. Otherwise, localize both $S$ and $R$ at their respective redundant homogeneous maximal ideals, say $S_+$ and $R_+$. Observe that $T(S_+) \subset R_+$. Hence, Theorem 3.0.1 applies with $\Delta = 0$ to give $s(R) = s(S)/d$.

[9] Say, Gorenstein rational ($F$-rational) 2-dimensional singularities.

[10] Using his terminology, except for $p = 2, 3, 5$, all rational double points are of classical form.

*Remark* 3.0.14. Notice that in the previous example, if $p$ divides $d$, then the extension is not generically separable, in fact, it ramifies everywhere. In case $d$ is prime-to-$p$, the extension is generically separable and $\mathrm{Tr}_{S/R} = d \cdot T$. In fact, dealing with this sort of phenomena where the extensions ramifies everywhere will be central in Chapter 5.

Throughout our discussion, we have seen how the surjectivity of $T$, $T(\mathfrak{n}) \subset \mathfrak{m}$, and the effectiveness of $\Delta^*$ are fundamental hypothesis in our proofs. The following examples demonstrate that we cannot weaken these three conditions on $T$ and $\Delta$.

**Example 3.0.15** (Necessity of the three hypothesis in Theorem 3.0.1)**.** In this example, we will consider $\Delta = 0$ throughout.

To see that the surjectivity of $T$ is necessary, we may consider [ST14, Example 7.12]. In this example, we are given with

$$R = \frac{\mathbb{F}_2[\![x,y,z]\!]}{\big(z^2+xyz+xy^2+x^2y\big)} \subset \frac{R[u,v]}{\big(u^2+xu+x, v^2+yv+y, z+xv+yu\big)} \cong \mathbb{F}_2[\![u,v]\!] = S$$

a quasi-étale[11] and degree 2 extension of 2-dimensional $\mathbb{F}_2$-algebras such that $\mathrm{Tr}_{S/R}$ is not surjective. In this example, $R$ is a log terminal singularity that is $F$-pure but not strongly $F$-regular. In fact, $R$ is the ring of invariants of $S$ under the action of $\mathbb{Z}/2\mathbb{Z} = \{0,1\}$ given by

$$1 \cdot u = u + \frac{1}{1+u}u^2, \quad 1 \cdot v = v + \frac{1}{1+v}v^2.$$

Moreover, one checks that $\mathrm{Tr}_{S/R}(u) = x$, $\mathrm{Tr}_{S/R}(v) = y$, and $\mathrm{Tr}_{S/R}(uv) = xy + z$. Then $\mathrm{Tr}_{S/R}(S) \subset (x,y,z)$. This example is due to M. Artin [Art75].

For the necessity of $T(\mathfrak{n}) \subset \mathfrak{m}$, consider any Noether normalization $R \subset S$ of a singular Gorenstein local ring $S$, *e.g.* $\mathscr{k}[\![x^2,y^2]\!] \subset \mathscr{k}[\![x^2,xy,y^2]\!]$, char $\mathscr{k} > 2$. Then $\omega_{S/R} \cong S$ for $S$ is Gorenstein, say $T$ is a free generator of $\omega_{S/R}$ as a $S$-module. Then $T$ is surjective, for $R \subset S$ must split, and $D_T = 0$. However, Theorem 3.0.1 fails for otherwise it would imply $s(S) = s(R) = 1$, but $R$ is singular. In the concrete example $\mathscr{k}[\![x^2,y^2]\!] \subset \mathscr{k}[\![x^2,xy,y^2]\!]$, we have that a free basis is $1, xy$. It is not difficult to see that the dual element of $xy$ in $\omega_{S/R}$, say $T$, is a free generator of $\omega_{S/R}$ as a $S$-module. By definition, it sends $xy$ to 1, so $T(\mathfrak{n}) \not\subset \mathfrak{m}$.

[11]That is, étale away from the closed point.

The same concrete example as above shows that the effectiveness of $\Delta^*$ is necessary, by considering $T_{S/R}$ instead. Indeed, $\mathrm{Tr}_{S/R} = 2xy \cdot T$, so that $\mathrm{Ram} = \operatorname{div} xy$. In this case, $\mathrm{Tr}_{S/R}$ is surjective and sends the maximal ideal into the maximal ideal, but the transformation rule does not hold.

**Example 3.0.16** (A Noether normalization)**.** In the previous example, we considered the Noether normalization $\mathscr{k}[\![x^2, y^2]\!] \subset \mathscr{k}[\![x^2, xy, y^2]\!]$. We noticed that we cannot apply Theorem 3.0.1 without divisors for this example. Nonetheless, if $\Delta = \frac{1}{2} \operatorname{div}_R x^2y^2$, then

$$f^*\Delta = \frac{1}{2} \operatorname{div}_S x^2y^2 = \frac{2}{2} \operatorname{div}_S xy = \mathrm{Ram}.$$

In other words, $\Delta^* = 0$ using the trace map as our global section of $\omega_{S/R}$. Therefore, by applying the transformation rule, we get that

$$s\Big(\mathscr{k}[\![x^2, xy, y^2]\!]\Big) = 2 \cdot s\Big([\![x^2, y^2]\!], \frac{1}{2} \operatorname{div} x^2y^2\Big) = 2 \cdot \frac{1}{4} = \frac{1}{2}$$

which coincided with our previous two computations in Example 3.0.12 and Example 3.0.13. The $F$-signature of the latter pair was computed directly in [BST12, Example 4.19].

To culminate this chapter, we discuss another aspect of the hypothesis of Theorem 3.0.1 in the generically separable case that will be useful later on. Recall that in the same setting of Theorem 3.0.1, we have that every $\varphi \in \mathscr{C}_e^{\Delta}$ has a unique transpose $\widehat{\varphi} \in \mathscr{C}_e^{\Delta^*}$ along $T$. That is, we have the following commutative diagram

$$\begin{array}{ccc} F^e_* f_* S & \xrightarrow{f_*\widehat{\varphi}} & f_* S \\ {\scriptstyle F^e_* T}\downarrow & & \downarrow{\scriptstyle T} \\ F^e_* R & \xrightarrow{\varphi} & R \end{array} \tag{3.8}$$

Suppose now that $(R, \Delta)$ is $F$-split. We claim that there exists $\varphi \in \mathscr{C}_e^{\Delta}$ surjective and $\mathfrak{m}$-compatible. Indeed, let $\psi \in \mathscr{C}_e^{\Delta}$ surjective (which exists since $(R, \Delta)$ is $F$-split). There exists a smallest $j > 0$ such that $\psi(F^e_*\mathfrak{m}^j) \subset \mathfrak{m}$. If $j = 1$, we may take $\varphi = \psi$. Otherwise, we have $\psi\big(F^e_*\mathfrak{m}^{j-1}\big) = R$ by hypothesis. Take $x \in \mathfrak{m}^{j-1}$ so that $\psi(F^e_* x) = 1$. Notice that $\varphi = \psi \cdot x$ has the required properties.

We notice that $\widehat{\varphi}$ is necessarily surjective and $\mathfrak{n}$-compatible, *c.f.* [Spe16, Theorem 3]. Indeed, the surjectivity was discussed right after Scholium 3.0.9. For the $\mathfrak{n}$-compatibility, if $\widehat{\varphi}(F^e_*\mathfrak{n}) = S$, then

$$T\big(\widehat{\varphi}(F^e_*\mathfrak{n})\big) = T(S) = R$$

however,

$$T\big(\widehat{\varphi}(F^e_*\mathfrak{n})\big) = \varphi\big(F^e_* T(\mathfrak{n})\big) \subset \varphi(F^e_*\mathfrak{m}) \subset \mathfrak{m}$$

which is a contradiction. It is worth noticing how both conditions on $T$ are utilized in the above argument.

In this way, we can mod out the bottom row of (3.8) by the compatible ideal $\mathfrak{m}$ and the top row by the compatible ideal $\mathfrak{n}$ to get the following commutative diagram at the residue field level

$$\begin{array}{ccc} F^e_*\ell & \longrightarrow & \ell \\ \downarrow & & \downarrow \\ F^e_*\mathscr{k} & \longrightarrow & \mathscr{k} \end{array}$$

Nonetheless, this is not quite useful in this generality. Let us assume now that $L/K$ is separable and $T = \mathrm{Tr}_{S/R}$ is the trace map. By [ST14, Proposition 5.6], we have that the diagram (3.8) implies

$$\begin{array}{ccc} F^e_* f_* S & \xrightarrow{f_*\widehat{\varphi}} & f_*S \\ \uparrow & & \uparrow \\ F^e_* R & \xrightarrow{\varphi} & R \end{array}$$

where the vertical arrows are the inclusion morphisms, which induces, after modding out by $\mathfrak{n}$ the top row and by $\mathfrak{m}$ the bottom row as above, the following commutative diagram on residue fields

$$\begin{array}{ccc} F^e_*\ell & \longrightarrow & \ell \\ \uparrow & & \uparrow \\ F^e_*\mathscr{k} & \longrightarrow & \mathscr{k} \end{array}$$

where the top arrow is surjective, hence nonzero. According to [ST14, Proposition 5.2], this implies that $\ell/\mathscr{k}$ is a separable field extension. We summarize our findings with the following proposition.

**Proposition 3.0.17** (Residual separability)**.** *Suppose we are in the same setting of Theorem 3.0.1 with $L/K$ separable and $T = \mathrm{Tr}_{S/R}$. If $(R, \Delta)$ is $F$-pure, then $\ell/\mathscr{k}$ as separable and moreover $[\ell : \mathscr{k}]$ divides $[L : K]$.*

*Proof.* It only remains to explain why $[\ell : \mathscr{k}]$ divides $[L : K]$. By passing to completions, we may assume that $R$ and $S$ are adically complete with respect to their maximal ideals. Therefore, we can choose coefficient fields $\mathscr{k} \subset R$ and $\mathscr{l} \subset S$. Moreover, since $\mathscr{l}/\mathscr{k}$ is separable, we can choose these coefficient fields in such a way that $\mathscr{k} \subset \mathscr{l}$. In this way, the extension $R \to S$ factors as

$$R \to R \otimes_{\mathscr{k}} \mathscr{l} \to S$$

where $R \to R \otimes_{\mathscr{k}} \mathscr{l}$ is $r \mapsto r \otimes 1$ and $R \otimes_{\mathscr{k}} \mathscr{l} \to S$ is given by $r \otimes l \mapsto rl$.

Observe that $R \otimes_{\mathscr{k}} \mathscr{l}$ is a finite extension of $R$ with unique maximal ideal $\mathfrak{m} \otimes_{\mathscr{k}} \mathscr{l}$. Furthermore, $R \otimes_{\mathscr{k}} \mathscr{l}$ is normal by [Sta18, Tag 06DF], for $\mathscr{l}/\mathscr{k}$ is geometrically normal.

On the other hand, $R \otimes_{\mathscr{k}} \mathscr{l} \to S$ is also finite and injective. Indeed, if it had a kernel, $S$ would be finite over a lower dimensional ring.

Let $E$ be the fraction field of $R \otimes_{\mathscr{k}} \mathscr{l}$. Then at the fraction field level, we would have inclusions $K \subset E \subset L$. However, $[E : K] = [\mathscr{l} : \mathscr{k}]$, for $R \otimes_{\mathscr{k}} \mathscr{l}$ is free of rank $[\mathscr{l} : \mathscr{k}]$ over $R$. This shows that $[\mathscr{l} : \mathscr{k}]$ divides $[L : K]$. ☕

The following easy observation will be useful and of interest later on.

**Proposition 3.0.18.** *Let $(R, \mathfrak{m}, \mathscr{k}) \subset (S, \mathfrak{n}, \mathscr{l})$ be a finite local extension of rings. Suppose there is $T \in \omega_{S/R}$ surjective and such that $T(\mathfrak{n}) \subset \mathfrak{m}$. If $\mathscr{l}/\mathscr{k}$ is trivial, then $T(1)$ is a unit in $R$. In particular, in case $R$ and $S$ are normal domains with separable extension of fraction fields $K \subset L$, if $\mathrm{Tr}_{S/R}$ is surjective and $\mathscr{k} = \mathscr{l}$, then $p$ does not divide $[L : K]$.*

*Proof.* The condition $T(\mathfrak{n}) \subset \mathfrak{n}$ just implies that $T\colon S \to R$ induces a quotient $\mathscr{k}$-linear map $\bar{T}\colon \mathscr{l} \to \mathscr{k}$. Notice that $\bar{T}$ inherits the surjectivity of $T$. Therefore, the extension $\mathscr{k} \subset \mathscr{l}$ being trivial just means that $\bar{T}$ is an isomorphism. This is equivalent to $\bar{T}(1) \neq 0$, *i.e.* $T(1) \not\equiv 0 \mod \mathfrak{m}$, so $T(1) \notin \mathfrak{m}$ and then $T(1)$ is a unit in $R$.

For the second part, we just note that the trace map always maps the maximal ideal into the maximal ideal; see for instance [Spe16, Lemma 9], and $\mathrm{Tr}_{S/R}(1) = [L : K]$. ☕

# Chapter 4

# Étale torsors and finiteness of the local étale fundamental group

In this chapter, we study the existence and measurement of nontrivial étale torsors over the regular locus of a strongly $F$-regular singularity. Although we are interested in studying general finite torsors, the étale case is interesting on its own, for in this case, we have the advantage of being able to measure the existence of such a torsors by using the local étale fundamental group, *i.e.* the étale fundamental group of the regular locus. In particular, by bounding the size of the local étale fundamental group in terms of intrinsic invariants of the singularity, we get a precise answer to what extent there are nontrivial étale torsors over the regular locus of a singularity.

Studying the finiteness of this fundamental group was motivated by the ground-breaking work by C. Xu on local étale fundamental groups of KLT singularities [Xu14]. In this paper, Xu partially answered a question by J. Kollár [Kol11, Question 26] on whether if $(0 \in X)$ is the germ of a KLT singularity, the (topological) fundamental group $\pi_1\big(X \smallsetminus \{0\}\big)$ is finite [Kol11, Question 26]. Xu answered this question by proving that the profinite completion of $\pi_1\big(X \smallsetminus \{0\}\big)$ is finite. This is the same as proving that the étale fundamental group $\pi_1^{\text{ét}}\big(X \smallsetminus \{0\}\big)$ is finite. Later, building on Xu's work, D. Greb, S. Kebekus, and T. Peternell in [GKP16, Theorem 1.13] proved the finiteness of the étale fundamental groups of the regular locus of KLT singularities; also see [TX17].

Given the strong relation between strongly $F$-regular singularities in positive characteristic and complex KLT singularities in characteristic zero, it is natural to ask whether the analogous result on local étale fundamental groups holds for strongly $F$-regular germs. In fact, this question was the genesis of this dissertation. For this, we have the following effective theorem bounding the order of the local étale fundamental group.

**Theorem 4.0.1** (On the finiteness of the local étale fundamental groups of strongly $F$-regular singularities)**.** *Let $(R, \mathfrak{m}, \mathscr{k}, K; \Delta)$ be a strongly $F$-regular germ*[1] *of prime characteristic $p > 0$ and dimension $d \geq 2$. Let $Z$ be a closed subscheme of $X = \operatorname{Spec} R$ of codimension at least 2 with complement $U$. Then the étale fundamental group of $U$ is finite with order at most $1/s(R, \Delta)$ and prime-to-$p$.*

*Equivalently, there is a generically Galois quasi-étale*[2] *cover $h^\star \colon Y^\star = \operatorname{Spec} S^\star \to X$ such that the open inclusion $V^\star := (h^\star)^{-1}(U) \to X^\star$ induces an isomorphism $\pi_1^{\text{ét}}(V^\star) \to \pi_1^{\text{ét}}(Y^\star)$, so that $\pi_1^{\text{ét}}(V^\star)$ is trivial for $S^\star$ is strictly local. In fact, $h^\star$ is étale over $U$ and its generic degree is at most $1/s(R)$ and prime-to-$p$.*

*Proof.* First of all, notice that since $R$ is a normal domain, we have that $U = X \setminus Z$ is connected and normal. Also, as customary for normal schemes, we choose the generic point as our base point of the fundamental group, *i.e.* our base point is going to be the field extension $K \to K^{\text{sep}}$, a separable closure of $K$. Then we have

$$\pi_1^{\text{ét}}(U) = \varprojlim \operatorname{Gal}(L/K)$$

where the inverse limit runs over the Galois category of all Galois finite extensions $K \subset L \subset K^{\text{sep}}$ such that the integral closure $S$ of $R$ in $L$ is étale over $U$, in particular quasi-étale and generically Galois. Additionally, since $R$ is strictly Henselian, we may consider $S$ to be (strictly) local, a normal germ of dimension $d$ and characteristic $p$ in fact. In particular, for any such extension

$$(R, \mathfrak{m}, \mathscr{k}, K; \Delta) \subset (S, \mathfrak{n}, \ell, L; f^*\Delta)$$

we have that $S \to \omega_{S/R}$ given by $1 \mapsto \operatorname{Tr}_{S/R}$ is an isomorphism of $S$-modules. Where $f \colon Y \to X$ is the corresponding morphism of schemes. Moreover, $\operatorname{Tr}_{S/R}$ would be forced to be surjective because $R \subset S$ splits given that $R$ is a splinter [HH94]. Alternatively, the surjectivity of the trace follows from [ST14, Theorem 7.6, Corollary 7.7]. Additionally, it is a well-known fact that $\operatorname{Tr}_{S/R}(\mathfrak{n}) \subset \mathfrak{m}$; see for example [Spe16, Lemma 9]. In other words, we are in the same setting of Theorem 3.0.1 and more specifically of Proposition 3.0.17. Therefore, $\ell/\mathscr{k}$ is separable and further trivial for $\mathscr{k}$ is separably closed (since $R$ is strictly Henselian).

[1] By a germ, we mean a strictly Henselian local domain (or simply a strictly local domain).
[2] Meaning étale in codimension 1.

By applying the Transformation rule for the $F$-signature, we then get

$$s(S, f^*\Delta) = [L : K] \cdot s(R, \Delta).$$

In particular, the $F$-signature goes up at least by a factor of 2 if the extension $R \subset S$ is not trivial. Therefore, any sequence of module-finite, local, generically Galois, and quasi-étale, extensions

$$(R, \mathfrak{m}, \mathcal{k}, K; \Delta) \subset (S_1, \mathfrak{n}_1, \mathcal{k}, L_1; \Delta_1) \subset (S_2, \mathfrak{n}_2, \mathcal{k}, L_2; \Delta_2) \subset \cdots$$

with $\Delta_{i+1}$ the pullback of $\Delta_i$ stabilizes, meaning that $S_i = S_{i+1}$ for $i$ sufficiently large. Indeed, since we started with a positive $F$-signature $s(R, \Delta)$, the sequence of $F$-signatures $s(S_i, \Delta_{i+1})$ would get arbitrarily large as $i$ increases if the above chain does not stabilize; contradicting the fact that $F$-signatures are at most 1 by definition.

The above shows that $\pi_1^{\text{ét}}(U)$ is finite. In this case, there will be an extension

$$(R, \mathfrak{m}, \mathcal{k}, K; \Delta) \subset (S^\star, \mathfrak{n}^\star, \mathcal{k}, L^\star; \Delta^\star)$$

representing the Galois category over which we calculate $\pi_1^{\text{ét}}(U)$. That is $\pi_1^{\text{ét}}(U) = \operatorname{Gal}(L^\star/K)$. Therefore,

$$\#\pi_1^{\text{ét}}(U) = [L^\star : K] = \frac{s(S^\star, \Delta^\star)}{s(R, \Delta)} \leq \frac{1}{s(R, \Delta)}$$

which proves the claimed upper bound on the order of $\pi_1^{\text{ét}}(U)$.

The statement that the order of $\pi_1^{\text{ét}}(U)$ is prime to the characteristic follows at once from Proposition 3.0.18. ☕

The following remarks about Theorem 4.0.1 are in order.

*Remark* 4.0.2 (Divisorial setup). Notice that the divisor $\Delta$ plays no role in the proof of Theorem 4.0.1. In particular, the order of $\pi_1^{\text{ét}}(U)$ is bounded by $1/s(R) \leq 1/s(R, \Delta)$ as well. Our reason for expressing the theorem including divisor is rather ideological. We are interested in studying pair singularities $(R, \Delta)$ rather than just singularities, so it is natural to us to express our results in this setup.

*Remark* 4.0.3 (Étale unipotent torsors). We would like to point out that the fact the order of $\pi_1^{\text{ét}}(U)$ is prime-to-$p$ is not a minor detail. Recall that in our proof, we said this is a direct consequence of Proposition 3.0.18. Let us take a closer look at this;

if we have that $(R, \mathfrak{m}, \mathscr{k}, K) = (S, \mathfrak{n}, \mathscr{k}, L)^G \subset (S, \mathfrak{n}, \mathscr{k}, L)$ is a $G$-torsor over $U$, then Proposition 3.0.18 implies that the order of $G$ is prime-to-$p$. In particular, if $G$ is a $p$-group, then any $G$-torsor over $U$ is necessarily trivial, meaning isomorphic to $U \times G$. We notice that the $p$-groups are exactly the étale unipotent group-schemes over $\mathscr{k}$. In Section 5.2, we will prove that this holds for general unipotent group-schemes over $\mathscr{k}$.

*Remark* 4.0.4 (Tame fundamental groups). We notice that the covers in our Galois category are all tamely ramified. In fact, they are cohomologically tamely ramified as defined in [KS10, CEPT96]. Precisely, a generically Galois and quasi-étale cover $R \subset S$ is cohomologically tamely ramified if the trace map $\mathrm{Tr}_{S/R} \colon S \to R$ is surjective. This is the strongest notion of tameness as proven in [KS10], including the one in [GM71]. Therefore, the étale fundamental group considered in this chapter coincides with any of the tame fundamental groups.

*Remark* 4.0.5 (Effectiveness of the finiteness). Notice that in contrast to Xu's result in the complex case, our result is effective in the sense that we show the local étale fundamental group is finite by bounding above its order by intrinsic invariants of the singularity.

Our bound on the order of $\pi_1^{\text{ét}}(U)$ by the reciprocal of the $F$-signature not only makes our result effective, but it is sharp too. Indeed, next two examples show that there are singularities in all dimensions for which $\#\pi_1^{\text{ét}}(U) = 1/s(R)$.

**Example 4.0.6** (Rational double points and sharpness of the upper bound)**.** Let us resume the Example 3.0.12. As Artin proved in [Art77, Corollary 2.7], if $p > 7$, we have that $\pi_1^{\text{ét}}(U) = G$ for a rational double point $R$, where $G \subset \mathrm{SL}_2$ is the subgroup realizing $R$ as a quotient of $S = \mathscr{k}[\![x, y]\!]$, *i.e.* $R = S^G$. Therefore,

$$\#\pi_1^{\text{ét}}(U) = \#G = 1/s(R)$$

so that the upper bound is attained in this example.

Furthermore, in this example, $S$ is the singularity $S^\star$ in Theorem 4.0.1. Additionally, it is worth mentioning that part of what Artin observed in [Art77] was that if $p > 7$, then $\#G$ is prime-to-$p$, and so $\pi_1^{\text{ét}}(U) = 1/s(R)$ was tame.

**Example 4.0.7** (Veronese subrings of the formal power series ring)**.** Let $S = S^\star = \mathscr{k}[\![x_1, ..., x_d]\!]$ and $R$ the $n$-th Veronese subring of $S$, *i.e.* $R = \mathscr{k}\left[\!\left[x_1^{a_1} \cdots x_d^{a_d} \;\middle|\; a_1 + \cdots +\right.\right.$

$a_d = n\rrbracket$. We have that $R$ is the ring of invariants of $S$ under the action of $\boldsymbol{\mu}_n$ given by $x_i \mapsto x_i \otimes \xi$. This extension $R \subset S$ is a $\boldsymbol{\mu}_n$-torsor away from the (singular) closed point of $X = \operatorname{Spec} R$. Therefore, if $n$ is a power of $p$, then the extension $R \subset S$ is radicial.[3] This implies that the homomorphism $\pi_1^{\text{ét}}\big(f^{-1}(U)\big) \to \pi_1^{\text{ét}}(U)$ induced by the pullback of $f \colon \operatorname{Spec} S \to X$ to $U$ is an isomorphism [Mur67, Proposition 7.2.2]. Nevertheless, $\pi_1^{\text{ét}}\big(f^{-1}(U)\big)$ is trivial by purity of the branch locus.

On the other hand, if $n$ is prime-to-$p$, we choose an isomorphism $\mathbb{Z}/n\mathbb{Z} \to \boldsymbol{\mu}_n(\Bbbk)$, which amounts to choose an $n$-th root of unity $\zeta \in \Bbbk$. In this way, $R$ is the ring of invariants of $S$ under the $\mathbb{Z}/n\mathbb{Z}$-action $x_i \mapsto \zeta \cdot x_i$. So the extension $R \subset S$ is Galois over $U$ with Galois group $\mathbb{Z}/n\mathbb{Z}$. Therefore, by purity of the branch locus, we get that $\pi_1^{\text{ét}}(U) \cong \mathbb{Z}/n\mathbb{Z}$, whose order equals $1/s(R)$.

The following example shows how Theorem 4.0.1 may fail for $F$-pure singularities.

**Example 4.0.8** (Failure of finiteness for $F$-pure germs)**.** Let $E$ be an ordinary elliptic curve over $\Bbbk = \Bbbk^{\text{alg}}$. It is well known that an elliptic curve is (globally) $F$-split if and only if it is ordinary; see [SZ15, Example 2.6]. For example, the elliptic curve given by $x^3 + y^3 + z^3 = 0$ is $F$-split if and only if $p \equiv 1 \mod 3$, using Fedder's criterion [Fed83, Lemma 1.6]; see [ST12, Exercise 2.6]. Then the affine cone over $E$ is an $F$-pure singularity. Now, consider the multiplication-by-$m$ isogeny on $E$, say $[m] \colon E \to E$, with $m$ prime-to-$p$. We have that $[m] \colon E \to E$ is a finite étale cover of $E$; see [Sil09] for further details. In fact, it is Galois with Galois group isomorphic to $\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$. These covers induced quasi-étale Galois covers on affine cones, in fact étale away from the origin. In particular, if $U$ is the affine cone of $E$ with the origin removed, we get that $\pi_1^{\text{ét}}(U)$ maps onto $\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$ for all $m$ prime-to-$p$, whereby it is not finite.

**Example 4.0.9** (Failure of finiteness for $F$-rational germs)**.** It is not expected that the local étale fundamental groups of $F$-rational germs are finite, for $F$-rational singularities are positive characteristic analogs of rational singularities in characteristic zero [Smi97]. In dimension 2, if the local étale fundamental group of a rational germ is finite, then it admits a quasi-étale cover by a germ whose fundamental group is trivial, and therefore a smooth germ by Mumford-Flenner's theorem [Mum61, Fle75]. Hence, the rational germ must be a log terminal singularity [Sch08].

[3] Meaning that $f \colon \operatorname{Spec} S \to \operatorname{Spec} R$ is universally injective; equivalently $f$ is injective as a map of topological spaces and all the residue field extensions are purely inseparable.

As an immediate corollary of Theorem 4.0.1, we obtain the following.

**Corollary 4.0.10** (Triviality of the local étale fundamental group for mild singularities)**.** *With the same setup as in Theorem 4.0.1, if* $s(R, \Delta) > 1/2$*, then* $\pi_1^{\text{ét}}(U)$ *is trivial.*

**Example 4.0.11** (Some singularities with $F$-signature more than 1/2)**.** The following are some examples of singularities $R$ with $s(R) > 1/2$. By [Sin05], we know that the $F$-signatures of the affine cones over $\mathbb{P}^1_{\Bbbk} \times \mathbb{P}^1_{\Bbbk}$, and over $\mathbb{P}^2_{\Bbbk} \times \mathbb{P}^2_{\Bbbk}$ are $4/3! = 2/3$ and $66/5! = 11/20$, respectively.

An important class of strongly $F$-regular rings are the determinantal rings over $\Bbbk$. For these, we have that the local étale fundamental group is trivial.

**Example 4.0.12** (Determinantal rings)**.** Let $S = \Bbbk[\![x_{i,j}]\!]$ be the formal power series ring in $m \times n$ variables. Let $I_t$ be the ideal generated by the $t \times t$ minors of the $m \times n$ matrix $[x_{i,j}]$, and let $R = S/I_t$, say $t \geq 2$. Let $x$ be the closed point of $X = \operatorname{Spec} R$, so that $X \smallsetminus \{x\}$ is regular, and in particular (locally) complete intersection. Following the terminology in [Cut95], we see that the difference between the dimension of $R$ and its deviation $\delta(R)$ is at least 3. Indeed, by denoting by $\mu(I_t)$ the minimal number of generators of $I_t$, we have

$$\begin{aligned} \dim R - \delta(R) = \dim R - \big(\mu(I_t) - (\dim S - \dim R)\big) &= \dim S - \mu(I_t) \\ &\geq mn - (mn - t^2 + 1) \\ &= t^2 - 1 \geq 3 \end{aligned}$$

where $\mu(I_t) \leq mn - t^2 + 1$ by [BV88, Corollary 5.21].[4] In other words, $\dim R \geq \delta(R) + 3 > \delta(R) + 2$. Therefore, one may use [Cut95, Corollary in page 175] to conclude that $\pi_1^{\text{ét}}(X \smallsetminus Z)$ is trivial for all closed subschemes $Z \subset X$ of codimension at least 3.

*Acknowledgement.* The author is greatly thankful to Jenny Kenkel for helping him to work out the example of determinantal rings.

[4] Indeed, since $I_t$ is prime, we have that its arithmetic rank equals $\mu(I_t)$.

# Chapter 5

# General finite torsors over strongly $F$-regular singularities

In this chapter, we study general finite torsors over strongly $F$-regular germs, focusing on the non-étale ones. To fix notation:

*Setup* 5.0.1. Let $(R, \mathfrak{m}, \Bbbk, K)$ be a strongly $F$-regular germ of prime characteristic $p > 0$ and dimension $d \geq 2$. Let $Z$ be a closed subscheme of $X = \operatorname{Spec} R$ of codimension at least 2 with open complement $U$. Let $I \subset R$ be the ideal of $R$ corresponding to $Z$.

In the étale case, our main result was the existence of a generically Galois cover $h^\star \colon Y^\star = \operatorname{Spec}\big(S^\star, \mathfrak{n}^\star, \Bbbk, L^\star\big) \to X$, étale over $U$, such that any étale torsor over $U^\star := (h^\star)^{-1}(U)$ is a torsor over $Y^\star$ and therefore trivial, additionally $[L^\star : K] \leq 1/s(R)$ and is prime-to-$p$.

From now onwards in this chapter, we assume $R$ is defined over $\Bbbk$, *i.e.* we suppose $R$ is a $\Bbbk$-rational germ, in such a way that all finite group-schemes are defined over $\Bbbk$. Additionally, we will suppose $\Bbbk$ is perfect (*i.e.* algebraically closed).

The ultimate goal in this chapter is to prove the existence of a "nice" cover $Y^\star \to X$ as above such that the restriction map of (isomorphims classes of) $G$-torsors

$$\varrho^1_{Y^\star}(G) \colon \check{H}^1\big(Y^\star_{\mathrm{ft}}, G\big) \to \check{H}^1\big(U^\star_{\mathrm{ft}}, G\big)$$

is surjective for all, or at least for a significant class of finite group-schemes over $\Bbbk$,[1] that is, so that every $G$-torsor over $U^\star$ extends across to a $G$-torsor over $Y^\star$. As opposed to the étale case, finite torsors over strictly local rings are no longer trivial. This is not a minor detail and is why we follow the above perspective.

[1]We will see that the largest class we are going to be able to handle is the class of finite group-schemes with either trigonalizable or nilpotent connected component at the identity.

Let us consider a couple of examples that are both motivating and clarifying. They support our conviction that non-étale torsors are fundamental in understanding singularities in positive characteristic.

**Example 5.0.2** (Rational double points in low characteristic and the local Nori fundamental group-scheme)**.** In [Art77], Artin was interested in the problem of whether for a surface singularity (in positive characteristic) $X$, the finiteness of the local étale fundamental group implies the existence of a cover $\hat{\mathbb{A}}^2_{\Bbbk} \to X$. He gave an affirmative answer for rational double points. The way he achieved this was by extending Lipman's classification [Lip69] of $E_8$ singularities in all characteristics to all rational double points, and explicitly constructing the cover $\hat{\mathbb{A}}^2_{\Bbbk} \to X$ for every $X$ in his classification. Of course, the characteristics of interest were $p = 2, 3, 5$.

More recently, H. Esnault and E. Viehweg [EV10], motivated by the failure of Mumford-Flenner theorem [Mum61, Fle75] in positive characteristic and the aforementioned work by Artin, introduced what they called the *local Nori fundamental group-scheme*. More precisely, to any $\Bbbk$-rational germ $X$, they associated a profinite group-scheme $\pi^{\mathrm{N}}_{\mathrm{loc}}(U, X, x)$, where $x$ represents the $\Bbbk$-rational point of $X$ and $U = X \smallsetminus \{x\}$. Roughly speaking, this fundamental group measures the finite torsors on $U$ that do not come from restricting a torsor over $X$.[2]

Esnault and Viehweg's main result [EV10, Theorem 4.2, Corollary 4.3] was that for a surface singularity $(X, x)$ over $\Bbbk$, the finiteness of $\pi^{\mathrm{N}}_{\mathrm{loc}}(U, X, x)$ implies that $(X, x)$ is rational. Furthermore, if $\pi^{\mathrm{N}}_{\mathrm{loc}}(U, X, x)$ is trivial, then $(X, x)$ is a rational double point. Then, they went through Artin's classification verifying that Artin's explicit cover $\hat{\mathbb{A}}^2_{\Bbbk} \to X$ was a (often non-étale) torsor over $U$ but not over $X$. This worked out for every member of Artin's list except possibly for three of them, namely $E_8^1$, $E_8^3$ in characteristic 2, and $E_8^1$ in characteristic 3. In conclusion, $\pi^{\mathrm{N}}_{\mathrm{loc}}(U, X, x)$ is not trivial except possibly for the aforementioned examples and the smooth germ. To the best of the author's knowledge, it does not follow from their work that $\pi^{\mathrm{N}}_{\mathrm{loc}}(U, X, x)$ is trivial even for $X = \hat{\mathbb{A}}^2_{\Bbbk}$. However, this is a direct application of A. Marrama's work on purity for torsors [Mar16].

**Example 5.0.3** (Radicial Veronese subrings of $S = \Bbbk[\![x_1, ..., x_d]\!]$)**.** Let us revisit Example 4.0.7. Using the same setting as before, consider the case of $n$ being a power

[2] Indeed, by the way it is defined, if $\pi^{\mathrm{N}}_{\mathrm{loc}}(U, X, x)$ is trivial, then $\varrho^1_X(G)\colon \check{H}^1(X_{\mathrm{ft}}, G) \to \check{H}^1(U_{\mathrm{ft}}, G)$ is surjective for all finite group-scheme $G$.

of $p$, say $q$. Then we have had that $\hat{\mathbb{A}}^d_{\Bbbk} \to X$ was a $\boldsymbol{\mu}_q$-torsor away from the $\Bbbk$-rational point of $X$. We will see later that this is the "nice" cover we are looking for.

In what follows, we intend to extend the methods used in Chapter 4 to the general case. The main result in Chapter 4, namely Theorem 4.0.1, was based on understanding the growth of the $F$-signature under quasi-étale local covers. For this, the transformation rule Theorem 3.0.1 played a primary role. We have seen in Example 3.0.15 how in order to apply the transformation rule, it is indispensable to have a map $T$ satisfying three fundamental properties. Seeking for this map and these properties will lead the rest of this chapter. Indeed, the present chapter is organized as follows:

- Suppose we have $R = S^G \subset S$ where $S$ is local ring and $G$ a finite group-scheme acting on $S$. If $G$ is étale, the trace map $\mathrm{Tr}_{S/R} \colon S \to R$ is given by the rule

$$\mathrm{Tr}_{S/R}(s) = \sum_{g \in G(\Bbbk)} g \cdot s.$$

  In particular, $\mathrm{Tr}_{S/R}$ is given purely in terms of $G$ and the action of $G$ on $S$. As general facts, $\mathrm{Tr}_{S/R}(\mathfrak{n}) \subset \mathfrak{n}$ [Spe16, Lemma 9], and $\mathrm{Tr}_{S/R}$ freely generates $\omega_{S/R}$ if and only if $R \subset S$ is Galois, hence étale, in codimension 1. The surjectivity of $\mathrm{Tr}_{S/R}$ is then forced by the strong $F$-regularity of $R$.

  In Section 5.1, we will generalize the afore-described picture to general finite group-schemes. We will apply the theory of integrals and traces often founded in the Hopf algebras literature. We will also prove the properties we need for these traces but could not find a reference for. Fundamentally, we will show that whether or not the trace is nonsingular characterizes torsorness. Its surjectivity will be inherited by $R$ once again. However, by looking at examples, we notice that the property $\mathrm{Tr}_{S/R}(\mathfrak{n}) \subset \mathfrak{m}$ does not hold in general, for example, if $G$ is unipotent.

- In Section 5.2, we analyze the case when $G$ is unipotent. We demonstrate that the map $\varrho^1_X(G) \colon \check{H}^1\big(X_{\mathrm{ft}}, G\big) \to \check{H}^1\big(U_{\mathrm{ft}}, G\big)$ is surjective for all unipotent group-schemes $G$ if $X$ is strongly $F$-regular (even just a splinter).

- In Section 5.3, we study the case $G$ is linearly reductive. In this case, we observed we can split in two more cases, namely, the case of Veronese-type and

Kummer-type cyclic covers. In both cases, $\mathrm{Tr}_{S/R}(\mathfrak{n}) \subset \mathfrak{m}$. However, only in the former case does the $F$-signature go up. Nevertheless, we argue why this is all we need.

- Finally, in Section 5.4, we put everything together to show the existence of a maximal cover that works for the class of group-schemes whose connected component at the identity is either trigonalizable or nilpotent.

## 5.1 Trace of quotients by finite group-schemes

We commence by showing how to construct the trace map $\mathrm{Tr}_{S/R}\colon S \to R$ of a quotient $R = S^G \subset S$ by the action of a finite group-scheme $G/\mathscr{k}$. After that, we will prove its relevant properties.

### 5.1.1 Construction of the trace

Before we start our discussion, we would like to recall the following terminology about bilinear and linear forms.

*Terminology* 5.1.1 (Linear and bilinear forms)*.* Let $M$ be a module over a ring $R$. We denote its dual $\mathrm{Hom}_R(M, R)$ by $M^\vee$. Note a *bilinear form* on $M$ over $R$ is the same as an element of $(M \otimes_R M)^\vee =: M^{\vee 2}$. By Hom-$\otimes$ adjointness, there are two natural isomorphisms $\upsilon_i\colon M^{\vee 2} \to \mathrm{Hom}_R(M, M^\vee)$, one per copy of $M$ in $M \otimes_R M$. A bilinear form $\Theta$ is *symmetric* if $\upsilon_1(\Theta) = \upsilon_2(\Theta)$; in that case, we write $\upsilon(\Theta)$ for either of these. A symmetric bilinear form $\Theta$ is said to be *nondegenerate* (resp. *nonsingular*)[3] if $\upsilon(\Theta)$ if injective (resp. an isomorphism).

If $M$ is free of finite rank, we have a determinant function $\det\colon M^{\vee 2} \to R$. We have that $\Theta$ is nondegenerate (resp. nonsingular) if and only if $\det \Theta$ is not a zero divisor (resp. a unit). In case $M$ is locally free of finite rank, we can associate to $\Theta$ a locally principal ideal $\det \Theta$ of $R$, *i.e.* an effective Cartier divisor on $\operatorname{Spec} R$, for naturally $(M^{\vee 2})_{\mathfrak{p}} \cong (M_{\mathfrak{p}})^{\vee 2}$.

Say $M =: S$ is an $R$-algebra, meaning there is a diagonal morphism $\Delta_{S/R}\colon S \otimes_R S \to S$. By taking its dual $\Delta_{S/R}^\vee\colon S^\vee \to (S \otimes_R S)^\vee$, we get a canonical way to obtain a bilinear form out of a linear form. We refer to $\theta \in S^\vee$ as nondegenerate or nonsingular

[3] Sometimes referred to as *unimodular* in the literature.

if $\Delta^{\vee}_{S/R}(\theta)$ is so. If $S$ is locally free of finite rank, one defines the *discriminant* of $\theta$ to be $\operatorname{disc}\theta := \det \Delta^{\vee}_{S/R}(\theta)$.

Observe that if the trace map $\operatorname{Tr}\colon S \to S^G$ exists, then they must exist particularly for the action of a finite group-scheme $G$ on itself, whose ring of invariants is the base field $\Bbbk$. We discuss these basic cases first, then the trace for any other quotient by $G$ is constructed from this one and the given action.

Thus, we want first to show the existence of a special $\Bbbk$-linear map $\operatorname{Tr}_{G/\Bbbk}\colon \mathscr{O}(G) \to \Bbbk$, or just $\operatorname{Tr}_G$ for short. It is not hard to see that $\mathscr{O}(G)$ is a Gorenstein $\Bbbk$-algebra and therefore, the $\mathscr{O}(G)$-module $\operatorname{Hom}_{\Bbbk}(\mathscr{O}(G), \Bbbk)$ is free of rank 1. We wish $\operatorname{Tr}_G$ to be a special generator of this Hom-set.

To see what this special generator is, note that $\mathscr{O}(G)$ coacts on itself via the coproduct $\nabla\colon \mathscr{O}(G) \to \mathscr{O}(G) \otimes \mathscr{O}(G)$, which means that $\mathscr{O}(G)^{\vee}$ acts on $\mathscr{O}(G)$, indeed $g \cdot \gamma = (\operatorname{id} \otimes g)\big(\nabla(\gamma)\big)$ for all $g \in \mathscr{O}(G)^{\vee}$, $\gamma \in \mathscr{O}(G)$. We want $\operatorname{Tr}_G \in \mathscr{O}(G)^{\vee}$ to yield invariants when it acts on elements via this action. That is, we want $\operatorname{Tr}_G \cdot \gamma$ to be an invariant element for all $\gamma \in \mathscr{O}(G)$. Thus, we require $g \cdot (\operatorname{Tr}_G \cdot \gamma) = g(1)(\operatorname{Tr}_G \cdot \gamma)$, for all $g \in \mathscr{O}(G)^{\vee}$, which leads to the desired property of

$$g \cdot \operatorname{Tr}_G = e^{\vee}(g) \cdot \operatorname{Tr}_G$$

for all $g \in \mathscr{O}(G)^{\vee}$.

Following Hopf algebras nomenclature, we are requiring $\operatorname{Tr}_G$ to be a *left integral* of the Hopf algebra $\mathscr{O}(G)^{\vee}$. To the best of the author's knowledge, this concept and its main properties were introduced by R. Larson and M. Sweedler in [LS69]. We summarize the definition, existence, and uniqueness of integrals for general Hopf algebras in the following theorem; for further details and proofs we recommend [Mon93, Chapter 2].

**Theorem 5.1.2.** ([Mon93, Definition 2.1.1, Theorem 2.1.3]) *Let $H$ be a Hopf algebra. We say that $t \in H$ is a* left integral *if $ht = e(h)t$ for all $h \in H$. Left integrals form a $\Bbbk$-submodule of $H$ denoted by $\int_H$. If $H$ is finite dimensional over $\Bbbk$, then $\int_H$ is unidimensional over $\Bbbk$.*

Thus, we take $\operatorname{Tr}_G$ to be any $\Bbbk$-generator of $\int_{\mathscr{O}(G)^{\vee}}$, which is unique up to scaling by elements of $\Bbbk^{\times}$. If there is $t \in \int_{\mathscr{O}(G)^{\vee}}$ such that $t(1) \neq 0$, then we always normalize to have $\operatorname{Tr}_G(1) = 1$. Maschke's theorem establishes this is the case exactly when $G$ is linearly reductive; see [Mon93, Theorems 2.2.1 and 2.4.6].

*Remark* 5.1.3. The following two remarks are in order.

(a) In order to be consistent with our forthcoming discussion, we should define $\mathrm{Tr}_G$ to be the $\mathcal{k}$-linear map $\gamma \mapsto t \cdot \gamma$ (for a nonzero choice of $t \in \int_{\mathcal{O}(G)^\vee}$) rather than $\gamma \mapsto t(\gamma)$. Nonetheless, these two are equivalent, for remarkably, $t \cdot \gamma = t(\gamma)$ for any left integral $t$, for $g\big(t \cdot \gamma\big) = g\big(t(\gamma)\big)$ for all $g \in \mathcal{O}(G)^\vee$. Indeed,

$$\begin{aligned} g\big(t \cdot \gamma\big) = g\Big((\mathrm{id} \otimes t)\big(\nabla(\gamma)\big)\Big) = (g \otimes t)\big(\nabla(\gamma)\big) = (g \cdot t)(\gamma) &= \big(g(1)t\big)(\gamma) \\ &= g(1)t(\gamma) \\ &= g\big(t(\gamma)\big). \end{aligned}$$

In other words, the following diagram is commutative

$$\begin{array}{ccc} \mathcal{O}(G) & \xrightarrow{\nabla} & \mathcal{O}(G) \otimes \mathcal{O}(G) \\ {\scriptstyle \mathrm{Tr}_G}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{Tr}_G} \\ \mathcal{k} & \xrightarrow{\quad u \quad} & \mathcal{O}(G) \end{array}$$

(b) $\mathrm{Tr}_G$ is nonsingular, see [Mon93, Theorem 2.1.3].

**Example 5.1.4** (Concrete examples of integrals)**.** Let $G$ be a finite discrete group. A left integral $t$ of $H := \mathrm{Hom}_{\mathsf{Set}}(G, \mathcal{k})$ must satisfy $\gamma t = \gamma(1)t$ for all $\gamma \in H$, in other words $\gamma(g)t(g) = \gamma(1)t(g)$ for all $\gamma \in H$, $g \in G$. Therefore, $\int_H = \mathcal{k} \cdot \varepsilon$, where $\varepsilon(g) = 0$ for all $g \neq 1$ and $\varepsilon(1) = 1$. Dually, a left integral $t$ of $\mathcal{k}[G]$ is characterized by $gt = t$ for all $g \in G$. For example, $t = \sum_{g \in G} g$ is a left integral, by uniqueness $\int_{\mathcal{k}[G]} = \mathcal{k} \cdot t$.

Consequently, for the constant group-scheme $G$, a trace $\mathrm{Tr}_G \colon \mathcal{O}(G) \to \mathcal{k}$ is given by $\gamma \mapsto \sum_{g \in G} \gamma(g)$. Notice $\mathrm{Tr}_G(1) = o(G)$, so if $p \neq o(G)$, we divide by this to have $\mathrm{Tr}_G(1) = 1$. On the other hand, for the diagonalizable group $D(G)$, its trace $\mathrm{Tr}_{D(G)} \colon \mathcal{k}[G] \to \mathcal{k}$ is given by projection onto the direct $\mathcal{k}$-summand generated by 1, so $\mathrm{Tr}_{k[G]}(1) = 1$. This coincides with the classical Reynold's operator.

For the unipotent $\boldsymbol{\alpha}_{p^e}$, a left integral $t$ has to satisfy $\xi^i \cdot t = 0$ for all $1 \leq i \leq p^e - 1$, for instance $t = \xi^{p^e-1}$, whereby $\int_{\boldsymbol{\alpha}_{p^e}} = \mathcal{k} \cdot \xi^{p^e-1}$. Hence, a trace for $\boldsymbol{\alpha}_{p^e}$ is obtained by projecting onto the direct $\mathcal{k}$-summand generated by $\xi^{p^e-1}$. In particular, $\mathrm{Tr}_{\boldsymbol{\alpha}_{p^e}}(1) = 0$.

Focusing on the affine case now, let $S$ be an algebra and consider an action $\alpha^{\#} \colon S \to \mathcal{O}(G_S)$ of $G$ on $\operatorname{Spec} S$, set $R = S^G$. Consider the $S$-linear map

$$\mathrm{Tr}_{G_S} := \mathrm{id} \otimes \mathrm{Tr}_G \colon \mathcal{O}(G_S) \to S$$

where the $S$-linear structure of $\mathscr{O}(G_S)$ is the one given by $u_S\colon S \to \mathscr{O}(G_S)$. By precomposing with $\alpha^\#$, one gets a $S$-linear map $\operatorname{Tr}_{G_S} \circ \alpha^\#\colon S \to S$. It is worth pointing out that this map is nothing but $s \mapsto \operatorname{Tr}_G \cdot s$, using the induced action of $\mathscr{O}(G)^\vee$ on $S$. The next proposition establishes that this map factors through the inclusion $R \subset S$. Thus, one defines $\operatorname{Tr}_{S/R}\colon S \to R$ to be the corresponding factor.

**Proposition 5.1.5.** *The $S$-linear map $\operatorname{Tr}_{G_S} \circ \alpha^\#\colon S \to S$ has image in $R$. One then defines $\operatorname{Tr}_{S/R}$ to be the restriction of the codomain.*

*Proof.* Recall this map is the same as $s \mapsto \operatorname{Tr}_G \cdot s$. It is straightforward to verify $\operatorname{Tr}_G \cdot s$ is an invariant under the action of $\mathscr{O}(G)^\vee$ on $S$, thereby it must be a coinvariant element under the coaction of $\mathscr{O}(G)$, as discussed in Remark 2.2.7. However, we would like to present a more direct proof that is inspired by the proof that faithfully flat extensions of rings are extensions of descent, as in [Mur67, Chapter 7]. Consider the following diagram:

$$\begin{tikzcd} S \arrow[r, "\alpha^\#"] & \mathscr{O}(G_S) \arrow[r, shift left, "\alpha^\# \otimes \mathrm{id}"] \arrow[r, shift right, "\nabla_S"'] \arrow[d, "\operatorname{Tr}_{G_S}"'] & \mathscr{O}(G_S) \otimes_S \mathscr{O}(G_S) \arrow[d, "\mathrm{id} \otimes \operatorname{Tr}_{G_S}"] \\ R \arrow[r, "\subset"] & S \arrow[r, shift left, "\alpha^\#"] \arrow[r, shift right, "u_S"'] & \mathscr{O}(G_S) \end{tikzcd}$$

Now, the bottom sequence is exact by definition. The top sequence, although not necessarily exact, satisfies $\alpha^\#(S) \subset \ker(\alpha^\# \otimes \mathrm{id}, \nabla_S)$, according to first axiom for $\alpha^\#$ to be a coaction. Thus, it suffices to prove the following two squares are commutative

$$\begin{tikzcd} \mathscr{O}(G_S) \arrow[r, "\alpha^\# \otimes \mathrm{id}"] \arrow[d, "\operatorname{Tr}_{G_S}"'] & \mathscr{O}(G_S) \otimes_S \mathscr{O}(G_S) \arrow[d, "\mathrm{id} \otimes \operatorname{Tr}_{G_S}"] \\ S \arrow[r, "\alpha^\#"] & \mathscr{O}(G_S) \end{tikzcd} \qquad \begin{tikzcd} \mathscr{O}(G_S) \arrow[r, "\nabla_S"] \arrow[d, "\operatorname{Tr}_{G_S}"'] & \mathscr{O}(G_S) \otimes_S \mathscr{O}(G_S) \arrow[d, "\mathrm{id} \otimes \operatorname{Tr}_{G_S}"] \\ S \arrow[r, "u_S"] & \mathscr{O}(G_S) \end{tikzcd}$$

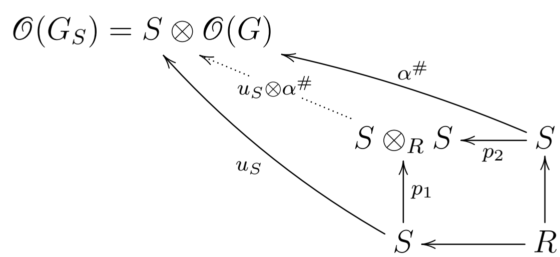
The commutativity of the first square is fairly straightforward to check. The commutativity of the second one is more interesting but follows from base changing by $S$ the commutativity square in Remark 5.1.3.

### 5.1.2 Initial properties of the trace

Let us begin by recalling the situation we want to generalize from the field-theoretic case: let $L/\mathcal{k}$ be a finite field extension and $G$ a finite discrete group acting (on the left) by $\Bbbk$-endomorphisms on $L$; if $K \subset L$ denotes the fixed subfield, then the extension $L/K$ is Galois with Galois groups $G$ if and only if $G$ acts faithfully on $L$. This at the same time is equivalent to $\mathrm{Tr}_{L/K}$ being nondegenerate and having $[L:K] = o(G)$. We get the following analog of this principle.

**Theorem 5.1.6** (Nonsingularity of $\mathrm{Tr}_{S/R}$ characterizes torsorness)**.** *Let $G$ be a finite group-scheme over $\Bbbk$. A finite $G$-quotient, say $q\colon \operatorname{Spec} S \to \operatorname{Spec} S^G$, is a $G$-torsor if and only if it is locally free of rank $o(G)$ and $\mathrm{Tr}_{S/R}$ is nonsingular.*

*Proof.* Let $R := S^G \subset S$, consider the fibered coproduct diagram




By definition, $R \subset S$ is a $G$-torsor if and only if it is faithfully flat and $\varphi := u_S \otimes \alpha^{\#}$ is an isomorphism. Notice that, in this case, we have that after a faithfully flat base change $R \subset S$ becomes $S \subset \mathscr{O}(G_S)$, which is a locally free extension of rank $o := o(G)$. Consequently, $R \subset S$ is locally free of rank $o$ too. Then, we can assume throughout that the extension in question is locally free of rank $o$. Assuming this, it remains to prove $\varphi$ is an isomorphism if and only if $\mathrm{Tr}_{S/R}$ is nonsingular. Notice both questions are local on $R$. In view of this, in the paragraph that follows, we explain how to reduce to the case $R$ is local.

Note that for $\mathfrak{p} \in \operatorname{Spec} R$, the localization of $\alpha^{\#}$ at $\mathfrak{p}$ induces a coaction $\alpha^{\#}_{\mathfrak{p}}\colon S_{\mathfrak{p}} \to S_{\mathfrak{p}} \otimes \mathscr{O}(G)$ of $\mathscr{O}(G)$ on $S_{\mathfrak{p}}$ whose ring of coinvariants is $R_{\mathfrak{p}}$. Moreover, we have $\varphi_{\mathfrak{p}} = u_{S_{\mathfrak{p}}} \otimes \alpha^{\#}_{\mathfrak{p}}$. On the other hand, the formula $\mathrm{Tr}_{S_{\mathfrak{p}}/R_{\mathfrak{p}}} = \mathrm{Tr}_{S/R} \otimes_R R_{\mathfrak{p}}$ also holds. In this manner, by localizing at prime ideals of $R$, we may assume that $R$ is local and so that $S$ is a semi-local free extension of $R$.

In order to show $\varphi\colon S \otimes_R S \to \mathscr{O}(G_S)$ is an isomorphism, it suffices to do it when considered as a $S$-linear map where the $S$-linear structures are given by $p_1$ and $u_S$, respectively. Let $s_1, ..., s_o$ be a basis of $S/R$ and let $\gamma_1, ..., \gamma_o$ be a basis of $\mathscr{O}(G)/\Bbbk$. Thus, $1 \otimes s_1, ..., 1 \otimes s_o$ is a basis for $p_1\colon S \to S \otimes_R S$ and similarly $1 \otimes \gamma_1, ..., 1 \otimes \gamma_o$ is a basis for $u_S\colon S \to \mathscr{O}(G_S)$. Set

$$\varphi(1 \otimes s_i) = (u_S \otimes \alpha^{\#})(1 \otimes s_i) = \alpha(s_i) = \sum_{m=1}^{o} a_i^m \otimes \gamma_m$$

that is, $M := (a_i^m)_{m,i}$ is the matrix representation of $\varphi$ in these bases. Thus, $\varphi$ is an isomorphism if and only if $M$ is nonsingular, *i.e.* $\det M \in S^{\times}$.

We proceed to describe now the symmetric $R$-matrix associated to $\mathrm{Tr}_{S/R}$ as $R$-bilinear form in terms of $M$ (a $S$-matrix). For this, we let $T := \big(\mathrm{Tr}_G(\gamma_m\gamma_n)\big)_{m,n}$ be the $\mathscr{k}$-matrix representing the $\mathscr{k}$-bilinear form $\mathrm{Tr}_G(- \cdot -)$ in the $\mathscr{k}$-basis $\gamma_1, \dots, \gamma_o$. Notice $T$ is nonsingular (aside of symmetric) by the second part of Remark 5.1.3. We then have:

**Claim 5.1.7.** *$M^{\top}TM$ is the matrix representation of $\mathrm{Tr}_{S/R}(- \cdot -)$ in the $R$-basis $s_1, \dots, s_o$.*

*Proof of claim.* This amounts to the following computation:

$$\begin{aligned}
\mathrm{Tr}_{S/R}(s_i \cdot s_j) = \mathrm{Tr}_{G_S}\big(\alpha^{\#}(s_i \cdot s_j)\big) &= \mathrm{Tr}_{G_S}\big(\alpha^{\#}(s_i) \cdot \alpha^{\#}(s_j)\big) \\
&= \mathrm{Tr}_{G_S}\left(\left(\sum_{m=1}^{o} a_i^m \otimes \gamma_m\right)\left(\sum_{n=1}^{o} a_j^n \otimes \gamma_n\right)\right) \\
&= \mathrm{Tr}_{G_S}\left(\sum_{1 \le m,n \le o} a_i^m a_j^n \otimes \gamma_m\gamma_n\right) \\
&= \sum_{1 \le m,n \le o} a_i^m a_j^n \cdot \mathrm{Tr}_G(\gamma_m\gamma_n) \\
&= \sum_{1 \le m,n \le o} a_i^m T_{mn} a_j^n = \big(M^{\top}TM\big)_{ij}.
\end{aligned}$$

☕

It is clear now that $\mathrm{Tr}_{S/R}$ is nonsingular if and only if $M$ is nonsingular. Indeed,

$$\operatorname{disc}\mathrm{Tr}_{S/R} = \det T \cdot (\det M)^2 = \operatorname{disc}\mathrm{Tr}_G \cdot (\det M)^2.$$

This proves the theorem. ☕

The following should be compared with [Mon93, Theorem 8.3.1] and her reference to the paper of H. F. Kreimer and M. Takeuchi, [KT81]. Although our proof is an elementary consequence of our proof of Theorem 5.1.6.

**Scholium 5.1.8.** *With the same setting as in Theorem 5.1.6 and its proof, suppose $R \subset S$ is locally free but $\varphi\colon S \otimes_R S \to \mathcal{O}(G_S)$ only surjective, then $\operatorname{Tr}_{S/R}$ is nondegenerate. If additionally $S$ is a domain, then $\varphi$ is an isomorphism and so $\operatorname{Tr}_{S/R}$ is nonsingular.*

*Proof.* Say $d$ is the rank of the extension $R \subset S$. From the surjectivity, we get $d \geq o$. So, we have that the matrix $M$ defines a surjective $S$-linear map $S^{\oplus d} \to S^{\oplus o}$; therefore, the $S$-linear transformation $M^\top : S^{\oplus o} \to S^{\oplus d}$ defined by $M^\top$ is injective, for this corresponds to the $S$-dual of the former. We claim now that the matrix $M^\top T M$ defines an injective $R$-linear operator $R^{\oplus o} \to R^{\oplus o}$. Indeed, we had already $M^\top$ and $T$ are injective, so it remains to see why $M$ is injective, more precisely, why if $M \cdot \vec{v} = 0$ for a column vector $\vec{v} \in R^{\oplus o}$, then $\vec{v} = 0$. This is just a different way to say $\alpha^\#$ is injective,[4] for if $\vec{v} = (r_1, ..., r_d)^\top$ and $M \cdot \vec{v} = (t_1, ..., t_o)^\top$, then $\alpha^\#(r_1 s_1 + \cdots + r_d s_d) = t_1 \otimes \gamma_1 + \cdots + t_o \otimes \gamma_o$. In other words, the determinant of $M^\top T M$ is not a zero divisor on $R$, which means $\operatorname{Tr}_{S/R}$ is nondegenerate.

For the final statement, if $S$ is further a domain, then the determinant of $M^\top T M$ would not be a zero divisor on $S$, as now being a zero divisor just means being zero. Therefore, $M^\top T M$ would also define an injective $S$-linear operator $S^{\oplus o} \to S^{\oplus o}$, which forces $M$ to be injective, *i.e.* $\varphi$ to be an isomorphism. ☕

The following corollary shows that the trace $\operatorname{Tr}_{S/R}$ just constructed satisfies the third hypothesis required to apply the transformation rule for the $F$-signature, namely the existence of an isomorphism $\tau\colon S \to \omega_{S/R}$.

**Corollary 5.1.9.** *If $R = S^G \subset S$ is a $G$-torsor, then $\operatorname{Tr}_{S/R}$ freely generates the $S$-module $\operatorname{Hom}_R(S, R) = \omega_{S/R}$. Furthermore, if $R$ and $S$ are both $\mathbf{S}_2$, this is the case even if $R \subset S$ is a $G$-torsor only in codimension-1, i.e. if $R_\mathfrak{p} \subset S_\mathfrak{p}$ is a $G$-torsor, by the induced action, for all height-1 prime ideals $\mathfrak{p}$ of $R$.*

*Proof.* The first statement is just a rephrasing of what it means for $\operatorname{Tr}_{S/R}$ to be nonsingular, *i.e.* the $S$-linear map $\tau\colon S \to \operatorname{Hom}_R(S, R)$ given by $s \mapsto \operatorname{Tr}_{S/R}(s \cdot -)$ is an isomorphism.

For the second statement, we just notice $S$ and $\omega_{S/R}$ are both $\mathbf{S}_2$ $R$-modules. Indeed, we have $S$ is $\mathbf{S}_2$ as $R$-module since restriction of scalars under finite maps does not change depth. For the $\mathbf{S}_2$-ness of $\omega_{S/R}$, we recommend the reader see [Sta18,

[4]Which at the same time followed from the second action axiom: $\mathrm{id}_S = e_S \circ \alpha^\#$.

Tag 0AUY]. Therefore, to check the aforementioned $S$-linear map $S \to \omega_{S/R}$ is an isomorphism, it suffices to do it in codimension-1 on $\operatorname{Spec} R$, which is be the case under this extra hypothesis. ☕

The next corollary is an analog of purity of branch locus for faithfully flat finite morphisms [AK70, Chapter VI, Theorem 6.8].

**Corollary 5.1.10.** *Suppose $Y = \operatorname{Spec} S \to X = \operatorname{Spec} S^G$ is locally free and a $G$-torsor in codimension-1, then it is a $G$-torsor everywhere.*

*Proof.* We have that the sheaf of principal ideals $\operatorname{disc} \operatorname{Tr}_{S/S^G}$ on $X$ determines the locus of points $x \in X$ where $Y \times_X \operatorname{Spec} \mathcal{O}_{X,x} \to \operatorname{Spec} \mathcal{O}_{X,x}$ is not a $G$-torsor under the induced action. In view of this, if the cover $Y \to X$ is not a torsor everywhere, then it fails to be so in codimension-1. ☕

This last corollary is analogous to the open nature of étaleness.

**Corollary 5.1.11** (Open nature of torsorness)**.** *Let $q \colon Y = \operatorname{Spec} S \to X = \operatorname{Spec} S^G$ be a $G$-quotient. The locus $W$ of points $x \in X$ where $q_x \colon Y \times_X \operatorname{Spec} \mathcal{O}_{X,x} \to \operatorname{Spec} \mathcal{O}_{X,x}$ is a torsor is Zariski open.*

*Proof.* Let $x \in W$. First of all, by the open nature of flatness, there is a Zariski open neighborhood $W'$ around $x$ such that $q_x$ is faithfully flat. Hence, the open $W' \setminus Z\big(\operatorname{disc} \operatorname{Tr}_{Y/X}\big) \ni x$ is contained in $W$. ☕

### 5.1.3 Cohomological tameness and total integrals

With the construction of the trace map $\operatorname{Tr}_{S/R} \colon S \to S^G$ in place, we are ready to formulate the notion of *tameness* our covers have. This will turn out to be a strong condition imposed by strong $F$-regularity, or more generally by splinters. Following [CEPT96], we pose the following definition

**Definition 5.1.12** (Cohomological tameness [CEPT96, KS10])**.** Let $G/\Bbbk$ be a finite group-scheme acting on a $\Bbbk$-algebra $S$ with corresponding ring of invariants $R \subset S$. We say that the extension $R \subset S$ is *(cohomologically) tame* if $\operatorname{Tr}_{S/R}$ is surjective.

In other words, we see that one of the three key hypotheses in Theorem 3.0.1 is really a tameness condition on the finite extension. In [KS10], several notions of tameness conditions are analyzed; however, cohomological tameness is the strongest one among them.

*Remark* 5.1.13 (Linearly reductive quotients are always tame). Notice that if $S \subset S^G = R$ is a $G$-quotient with $G$ linearly reductive, then it is automatically tame. Indeed, by Maschke's theorem, we have that the integral of $G$ satisfies $\mathrm{Tr}_G(1) = 1$, then

$$\mathrm{Tr}_{S/R}(1) = \mathrm{Tr}_{G_S}\big(\alpha^{\#}(1)\big) = \mathrm{Tr}_G(1) = 1$$

so that $\mathrm{Tr}_{S/R}$ is a splitting and therefore surjective. In the opposite case, if $G$ is unipotent, then $S^G \subset S$ is tame only if it is a trivial torsor [CEPT96, Proposition 6.2].

*Remark* 5.1.14 (Total integrals). In the Hopf algebras literature, the surjectivity of the trace $\mathrm{Tr}_{S/S^G} \colon S \to S^G$ is referred to as the existence of *total integrals* for the right $\mathscr{O}(G)$-comodule algebra $S$. To the best of the author's knowledge, the theory of total integrals was introduced in the work of Y. Doi [Doi85]. However, it was formulated in a slightly different language. Nonetheless, a complete proof of the equivalence between the existence of Doi's total integrals and the surjectivity of the trace appeared in [CF92]. For further details, see [Mon93, §4.3].

Thus, we can see how splinters and therefore strongly $F$-regular rings impose strong conditions on finite quasi-torsors[5] over them. Concretely,

**Proposition 5.1.15.** *Let $R \subset S$ be a quotient by the action of a finite group-scheme $G/\Bbbk$. Suppose $R$ is a splinter $\mathbf{S}_2$ domain and $S$ is $\mathbf{S}_2$. If $R \subset S$ is a $G$-torsor in codimension 1, then $R \subset S$ is tame.*

*Proof.* First of all, by Corollary 5.1.9, we have that $\mathrm{Tr}_{S/R}$ generates $\omega_{S/R}$ as an $S$-module. On the other hand, since $R$ is a splinter, there must exist a splitting $S \to R$. Then, an $S$-multiple of $\mathrm{Tr}_{S/R}$ sends 1 to 1, therefore $\mathrm{Tr}_{S/R}$ is surjective. ☕

### 5.1.4 The leading question

So far, the trace map $\mathrm{Tr}_{S/R} \colon S \to S^G$ associated to a $G$-quotient has had all the good properties the classic trace[6] has. Nonetheless, to our surprise, it may happen that $\mathrm{Tr}_{S/R}(\mathfrak{n}) \not\subset \mathfrak{m}$. The following example provides cases of this.

[5] Meaning torsor in codimension 1.

[6] Meaning the trace of a generically separable extension.

**Example 5.1.16** (Failure of $\mathrm{Tr}_{S/R}(\mathfrak{n}) \subset \mathfrak{m}$)**.** Consider $S = R[t]/\big(t^p - r\big)$. Let us also consider in this example the possibility that $\mathscr{k}$ may not be algebraically closed. Note that $S$ is local for all $r$. However, what its maximal ideal $\mathfrak{n} \leftrightarrow y$ is depends on $r$. Indeed, if $r \in \mathfrak{m}$, then $\mathfrak{n} = \mathfrak{m} \oplus R \cdot t \oplus \cdots \oplus R \cdot t^{p-1}$, in particular $y$ is a $\mathscr{k}$-rational point lying over $x$. Suppose now $r \notin \mathfrak{m}$; we have two cases depending on whether or not $r$ has a $p$-th root residually. If $r = u^p + x$ for some $u \in R^\times$ and $x \in \mathfrak{m}$, then $\mathfrak{n} = \mathfrak{m}S + (t-u)$; in this case, $y$ is a $\mathscr{k}$-rational point too. But if $r$ has no $p$-th roots even residually, then $\mathfrak{n} = \mathfrak{m}S$. However, it would be impossible if we demand $y$ to be a $\mathscr{k}$-rational point, for at the residue field level, we would have $\mathscr{k} \subset \mathscr{k}\big(r^{1/p}\big)$.

Now, $(R,\mathfrak{m}) \subset (S,\mathfrak{n})$ is an $\boldsymbol{\alpha}_p$-torsor for all $r \in R$, via the coaction $\alpha^{\#} \colon t \mapsto t \otimes 1 + 1 \otimes \xi$. If $r \in \mathfrak{m}$, then as we had seen above $t^{p-1} \in \mathfrak{n}$, but

$$\begin{aligned}\mathrm{Tr}_{S/R}(t^{p-1}) = \big(\mathrm{id} \otimes \mathrm{Tr}_{\boldsymbol{\alpha}_p}\big)\big(\alpha^{\#}(t^{p-1})\big) &= \big(\mathrm{id} \otimes \mathrm{Tr}_{\boldsymbol{\alpha}_p}\big)\big((t \otimes 1 + 1 \otimes \xi)^{p-1}\big) \\ &= \big(\mathrm{id} \otimes \mathrm{Tr}_{\boldsymbol{\alpha}_p}\big)\left(\sum_{i=0}^{p-1} \binom{p-1}{i} t^{p-1-i} \otimes \xi^i\right) \\ &= \sum_{i=0}^{p-1} \binom{p-1}{i} t^{p-1-i}\, \mathrm{Tr}_{\boldsymbol{\alpha}_p}(\xi^i) = 1\end{aligned}$$

see Example 5.1.4. Hence, we cannot expect in general the trace to map the maximal ideal into the maximal ideal.

The same phenomena $\mathrm{Tr}_{S/R}$ can also happen even for $\boldsymbol{\mu}_p$-torsors. Indeed, if $r$ is a unit, then $(R,\mathfrak{m}) \subset (S,\mathfrak{n})$ is a $\boldsymbol{\mu}_p$-torsor under the coaction $t \mapsto t \otimes \zeta$. If $r = u^p + x$ as above, then $t - u \in \mathfrak{n}$ but $\mathrm{Tr}_{S/R}(t-u) = u$, by a similar computation as the one above. Amusingly, in case $r$ has no $p$-th roots even residually, we have $\mathrm{Tr}_{S/R}(\mathfrak{n}) \subset \mathfrak{m}$, and the transformation rule takes the form $p \cdot s(S) = p \cdot s(R)$, so $s(S) = s(R)$. In view of this, one may ask whether if $r = u^p + x$, with $x \neq 0$, there is any chance that $s(S)$ is at least $s(R)$. The following example contradicts this. Let $p = 3$, $R = \mathscr{k}\llbracket s, x^3 \rrbracket$ and $r = 1 + x^6$, then $S = \mathscr{k}\llbracket s, x^2, x^3 \rrbracket$, which is not even normal.[7]

*Remark* 5.1.17. Notice that in all the examples above Example 5.1.16, we had that the cover $R \subset S$ was a torsor everywhere. It is then natural to ask the following.

*Question* 5.1.18. Let $(R,\mathfrak{m},\mathscr{k}) \subset (S,\mathfrak{n},\ell = \mathscr{k})$ be a local $G$-quotient that is a $G$-torsor in codimension 1 but not everywhere; is $\mathrm{Tr}_{S/R}(\mathfrak{n})$ contained in $\mathfrak{m}$?

[7]The extra variable "$s$" is just for $R$ to be bidimensional.

Investigating this question will dominate the remainder of this chapter. For this, we will consider separately the unipotent and linearly reductive cases. These would handle the general abelian case, or more generally, the trigonalizable case.

## 5.2 The case of unipotent torsors

Our main result in this section establishes that Question 5.1.18 is vacuous if $G$ is unipotent and the ring of invariants $R$ is strongly $F$-regular. More generally, we will show that Question 5.1.18 is vacuous if $G$ is unipotent and the $G$-quotient $R \subset S$ is cohomologically tamely ramified, a condition guaranteed by the strong $F$-regularity of $R$.

**Theorem 5.2.1.** *With notation as in Setup 5.0.1, every unipotent $G$-torsor over $U$ comes from restricting a $G$-torsor over $X$. That is, the restriction map of torsors*

$$\varrho^1_X(G) \colon \check{H}^1\big(X_{\mathrm{ft}}, G\big) \to \check{H}^1\big(U_{\mathrm{ft}}, G\big)$$

*is surjective for all unipotent group-schemes $G/\Bbbk$.*

We will provide two proofs of this theorem. The first one is an application of the work in [CEPT96], and hence it is shorter looking. The second proof was our original approach, and is quite direct. We consider the techniques involved in our proof to be quite valuable and interesting in their own right, so much so that we recycle them into the proof of our main result in Section 5.4.

*First proof of Theorem 5.2.1.* It is established in [CEPT96, Proposition 6.2] that if $R \subset S$ is a tame $G$-quotient by a unipotent group-scheme $G/\Bbbk$, then the extension must be a torsor (everywhere). Therefore, the result follows from Proposition 2.3.5 and Proposition 5.1.15. ☕

*Remark* 5.2.2. Notice that this proof works for $X$ just a splinter.

The second proof will be a little journey, so we will need some preparatory discussion. First, remember all unipotent group-schemes admit a central normal series whose quotients are (isomorphic to) subgroups of $\mathbb{G}_{\mathrm{a}}$, thus *elementary unipotent group-schemes* following the terminology of [Mil17] are in particular abelian. In view of this, we will show Theorem 5.2.1 first in the abelian case and the general case is obtained from this one by induction on the order.

### 5.2.1 Elementary case

In case $G$ is abelian, the isomorphism classes of $G$-torsors over a scheme $Y$ are naturally classified by $H^1(Y_{\mathrm{ft}}, G)$, the derived-functor of flat cohomology. If further $I = \mathfrak{m}$ (*i.e.* $U$ is the punctured spectrum), we have the short exact sequence from [Bou78, Corollaire 4.9, Chapitre III],

$$0 \to H^1(X_{\mathrm{ft}}, G) \to H^1(U_{\mathrm{ft}}, G) \to \mathrm{Hom}\big(G^\vee, \mathrm{Pic}_{R/\Bbbk}(U)\big) \to 0.$$

Hence, every abelian $G$-torsor over $U = \mathrm{Spec}^\circ R$ extends across to a $G$-torsor over $X$ if and only if the abelian group $\mathrm{Hom}(G^\vee, \mathrm{Pic}_{R/\Bbbk}(U))$ is trivial. We would like to use this to simplify our forthcoming arguments. However, to the best of the author's knowledge, it is unknown whether Boutot's theory of the local Picard scheme and his short exact sequence extend to general $I$ of height at least 2. It is worth recalling that this is very limiting for us, firstly because we are interested in obtaining potential global results like in [BCRG+17], secondly, the case $I = \mathfrak{m}$ is most interesting only for surfaces singularities; we are, however, interested in higher dimensions.

To bypass this issue, we take a closer look at Boutot's arguments in [Bou78, Chapitre III] to see what information about the cokernel of $\rho^1_X(G)\colon H^1(X_{\mathrm{ft}}, G) \to H^1(U_{\mathrm{ft}}, G)$ we can obtain. Denoting this cokernel by $\mathrm{Ob}_X(G)$, we have

**Lemma 5.2.3.** *Let* $* \to G' \to G \to G'' \to *$ *be a short exact sequence of abelian group-schemes with* $\mathrm{Ob}_X(G') = 0 = \mathrm{Ob}_X(G'')$*. Then* $\mathrm{Ob}_X(G) = 0$*.*

*Proof.* Recall $\rho^i_X(-)\colon H^i(X_{\mathrm{ft}}, -) \to H^i(U_{\mathrm{ft}}, -)$ are obtained as the left-derived natural transformations of $\Gamma(X, -) \to \Gamma(U, -)$, so compatible with the $\delta$-structures. We then have the following commutative and horizontally exact diagram:

$$\begin{array}{ccccccccc} H^0(X_{\mathrm{ft}}, G'') & \xrightarrow{\delta} & H^1(X_{\mathrm{ft}}, G') & \longrightarrow & H^1(X_{\mathrm{ft}}, G) & \longrightarrow & H^1(X_{\mathrm{ft}}, G'') & \xrightarrow{\delta} & H^2(X_{\mathrm{ft}}, G') \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^0(U_{\mathrm{ft}}, G'') & \xrightarrow{\delta} & H^1(U_{\mathrm{ft}}, G') & \longrightarrow & H^1(U_{\mathrm{ft}}, G) & \longrightarrow & H^1(U_{\mathrm{ft}}, G'') & \xrightarrow{\delta} & H^2(U_{\mathrm{ft}}, G') \end{array}$$

Our hypothesis is that second and fourth vertical arrows are surjective. Therefore, according to the 5-lemma, to get surjectivity of the third one, we need the fifth arrow to be injective. However, Boutot does show [Bou78, Chapitre III, Corollaire 4.9] that $H^2(X_{\mathrm{ft}}, G') = 0$ for all abelian $G'$. In fact, all cohomologies higher than 2 vanish; see [Bha12, Proposition 3.1] for a nice, conceptual proof. ☕

The following proposition demonstrates Theorem 5.2.1 in the abelian case.

**Proposition 5.2.4.** *If $G$ is an abelian unipotent group-scheme, then* $\mathrm{Ob}_X(G) = 0$.

*Proof.* In view of Lemma 5.2.3, we may assume $G$ is simple. That is, it suffices to treat the cases $G = \mathbb{Z}/p\mathbb{Z}$ and $G = \boldsymbol{\alpha}_p$.

**Claim 5.2.5.** $\mathrm{Ob}_X(\mathbb{Z}/p\mathbb{Z}) = 0$.

*Proof of claim.* This is a consequence of Artin–Schreier Theory; see [Mil80, Chapter III, Proposition 4.12]. According to this, from the long exact sequence on flat cohomology derived from (2.3), we get

$$\mathrm{Ob}_X(\mathbb{Z}/p\mathbb{Z}) \cong H^1(U, \mathscr{O}_U)^F := \big\{a \in H^1(U, \mathscr{O}_U) = H^2_I(R) \mid Fa = a\big\}.$$

This happens to be zero if the ring $R$ is just $F$-rational and $I = \mathfrak{m}$, for stable elements under the action of Frobenius must be zero; see [Smi97, §2, Theorem 2.6]. For our general $I$ and $R$ strongly $F$-regular, one proves $H^1(U, \mathscr{O}_U)^F$ is zero as follows. Take $a \in H^1(U, \mathscr{O}_U)^F$ and let $r$ be a nonzero element in the annihilator of $a$.[8] Let $\varphi \in \mathrm{Hom}_R(F^e_* R, R)$ such that $\varphi(F^e_* r) = 1$, *i.e.* $\varphi$ splits the $R$-linear composite

$$R \to F^e_* R \xrightarrow{\cdot r} F^e_* R.$$

By applying the local cohomology functor $H^2_I(-)$, we get that $\phi := H^2_I(\varphi)$ splits the composite

$$H^2_I(R) \xrightarrow{F^e} H^2_I(R) \xrightarrow{\cdot r} H^2_I(R)$$

therefore $a = \phi\big(r \cdot F^e a\big) = \phi(r \cdot a) = \phi(0) = 0$. This proves the claim. ☕

**Claim 5.2.6.** $\mathrm{Ob}_X(\boldsymbol{\alpha}_p) = 0$.

*Proof.* As before, from the long exact sequence on flat cohomology derived from (2.2),

$$\mathrm{Ob}_X(\boldsymbol{\alpha}_p) \cong \ker\big(H^1(U, \mathscr{O}_U) \xrightarrow{F} H^1(U, \mathscr{O}_U)\big).$$

This kernel is zero by definition for $F$-injective $X$ and $I = \mathfrak{m}$. For general $I$, one can use $F$-purity to show $H^2_I(R) \xrightarrow{F} H^2_I(R)$ is injective. Indeed, if $\varphi$ splits $R \to F^e_* R$, then $H^2_I(\varphi)$ splits $H^2_I(R) \xrightarrow{F} H^2_I(R)$, forcing it to be injective. This proves the claim. ☕

[8] Recall that every element of $H^i_I(R)$ is annihilated by some power of $I$.

Then the proposition holds.

*Remark* 5.2.7. Claim 5.2.5 follows for $X$ a splinter from Proposition 3.0.18. This gives an alternate demonstration.

*Acknowledgement.* The author would like to thank Christian Liedtke, who, to our knowledge, first observed the results of Claim 5.2.5 and Claim 5.2.6 and made us aware of them. These results will be treated in an upcoming preprint by Christian Liedtke and Gebhard Martin [LM17].

### 5.2.2 General case

To handle the general case, we proceed by induction on the order of the group-scheme along with the fact it admits a central elementary (necessarily) unipotent subgroup whose quotient is (necessarily) unipotent. However, we shall require the use of nonabelian first and second flat cohomology as treated in [Gir71]. For sake of notation, we denote this cohomology over a scheme $Y$ by $\check{H}^i(Y_{\mathrm{ft}}, G)$, $i = 1, 2$. However, it coincides with the derived-functor flat cohomology if $G$ is a sheaf of abelian groups.

*Second proof of Theorem 5.2.1.* Let $\varrho^1_X(G)\colon \check{H}^1(X_{\mathrm{ft}}, G) \to \check{H}^1(U_{\mathrm{ft}}, G)$ denote the restriction map (of pointed sets). We prove it is surjective if $G$ is unipotent. Let $G'$ be a nontrivial central (so normal and abelian) subgroup of $G$ with corresponding short exact sequence

$$* \to G' \to G \to G'' \to *$$

so that $G''$ is unipotent and $o(G'') < o(G)$. We may also assume $o(G') < o(G)$, otherwise we are done by Section 5.2.1. Consider now the commutative digram

$$\begin{array}{ccccccccc}
H^0(X_{\mathrm{ft}}, G'') & \longrightarrow & \check{H}^1(X_{\mathrm{ft}}, G') & \longrightarrow & \check{H}^1(X_{\mathrm{ft}}, G) & \longrightarrow & \check{H}^1(X_{\mathrm{ft}}, G'') & \longrightarrow & \check{H}^2(X_{\mathrm{ft}}, G') \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
H^0(U_{\mathrm{ft}}, G'') & \longrightarrow & \check{H}^1(U_{\mathrm{ft}}, G') & \longrightarrow & \check{H}^1(U_{\mathrm{ft}}, G) & \longrightarrow & \check{H}^1(U_{\mathrm{ft}}, G'') & \longrightarrow & \check{H}^2(U_{\mathrm{ft}}, G')
\end{array}$$

where the horizontal sequences are exact sequences of pointed sets.

Notice that $\check{H}^2(X_{\mathrm{ft}}, G') = H^2(X_{\mathrm{ft}}, G') = 0$, as before, for $G'$ is abelian unipotent. That is, $\check{H}^2(X_{\mathrm{ft}}, G')$ is a singleton. We have that first and third vertical arrows are onto by the inductive hypotheses. Unfortunately, we cannot apply the 5-lemma to get the surjectivity of the middle one as the sets in consideration are no longer groups. Let us chase the diagram; this will inspire a strategy to go around it. Let $t_0 \in \check{H}^1(U_{\mathrm{ft}}, G)$. It maps to $t_1 \in \check{H}^1(U_{\mathrm{ft}}, G'')$, which extends across to $t_2 \in \check{H}^1(X_{\mathrm{ft}}, G'')$. However, this lifts to $t_3 \in \check{H}^1(X_{\mathrm{ft}}, G)$ as $\check{H}^2(X_{\mathrm{ft}}, G')$ is trivial. Let $t_4 \in \check{H}^1(U_{\mathrm{ft}}, G)$ be the restriction of $t_3$ to $U$. At this point, we would like to substract $t_4$ from $t_0$ as $t_4 \mapsto t_1$. However, this does not make sense in this setting. Fortunately, we may make sense of this by *changing the origin via twisted forms* as in [Gir71, Chapitre III, §2, 2.6]. Indeed, we have the conjugate representation $G \to \operatorname{Aut} G$ of $G$, defined by the action of $G$ on itself by inner automorphisms. This gives a map of pointed sets $\check{H}^1(U_{\mathrm{ft}}, G) \to \check{H}^1(U_{\mathrm{ft}}, \operatorname{Aut} G)$, where $\check{H}^1(U_{\mathrm{ft}}, \operatorname{Aut} G)$ classifies the so-called twisted forms of $G$; see [Gir71, Chapitre III, §2, 2.5]. For sake of notation, we write $t \mapsto {}^tG$ for such a map realizing $G$-torsors as twisted forms of $G$. We have a bijection:

$$\theta_t : \check{H}^1(U_{\mathrm{ft}}, G) \to \check{H}^1(U_{\mathrm{ft}}, {}^tG)$$

where $t$ gets mapped to the trivial class in $\check{H}^1(U_{\mathrm{ft}}, {}^tG)$. Moreover, if $G$ happens to be abelian, this map is nothing but $t' \mapsto t' - t$. See [Gir71, Chapitre III, Remarque 2.6.3].

Thus, it is clear that what we need to do is to twist the short exact sequence

$$* \to G' \to G \to G'' \to *$$

by the $G$-torsor $t_4$. Precisely, $G$ acts by inner automorphism on both $G'$ and $G''$, then $t_4$ can be realized as twisted forms of both $G'$ and $G''$. Since $G'$ is central, its twisted form yielded by $t_4$ is the trivial one, namely $G$ itself. On the other hand, the twisted form of $G''$ that $t_4$ gives is ${}^{t_1}G''$. Summing up, we have a short exact sequence on $U$

$$* \to G' \to {}^{t_4}G \to {}^{t_1}G'' \to *$$

and a commutative diagram

$$\begin{array}{ccccc} H^1(U_{\mathrm{ft}}, G') & \longrightarrow & \check{H}^1(U_{\mathrm{ft}}, G) & \longrightarrow & \check{H}^1(U_{\mathrm{ft}}, G'') \\ & & \downarrow{\scriptstyle \theta_{t_4}} & & \downarrow{\scriptstyle \theta_{t_1}} \\ H^1(U_{\mathrm{ft}}, G') & \longrightarrow & \check{H}^1(U_{\mathrm{ft}}, {}^{t_4}G) & \longrightarrow & \check{H}^1(U_{\mathrm{ft}}, {}^{t_1}G'') \end{array} \tag{5.1}$$

see [Gir71, Chapitre III, §3, Corollaire 3.3.5]. In the same way, one can twist

$$* \to G' \to G \to G'' \to *$$

by $t_3$, obtaining a short exact sequence

$$* \to G' \to {}^{t_3}G \to {}^{t_2}G'' \to *$$

on $X$. Furthermore, we get the commutative diagram,

$$\begin{array}{ccccc} H^1(X_{\mathrm{ft}}, G') & \longrightarrow & \check{H}^1(X_{\mathrm{ft}}, {}^{t_3}G) & \longrightarrow & \check{H}^1(X_{\mathrm{ft}}, {}^{t_2}G'') \\ \downarrow & & \downarrow & & \downarrow \\ H^1(U_{\mathrm{ft}}, G') & \longrightarrow & \check{H}^1(U_{\mathrm{ft}}, {}^{t_4}G) & \longrightarrow & \check{H}^1(U_{\mathrm{ft}}, {}^{t_1}G'') \end{array}$$

Now, we can take $\theta_{t_4}(t_0)$ in $\check{H}^1(U_{\mathrm{ft}}, {}^{t_4}G)$. Notice that by the commutative diagram (5.1), the torsor $\theta_{t_4}(t_0)$ is mapped to $\theta_{t_1}(t_1)$ under $\check{H}^1(U_{\mathrm{ft}}, {}^{t_4}G) \to \check{H}^1(U_{\mathrm{ft}}, {}^{t_1}G'')$, *i.e.* $\theta_{t_4}(t_0)$ is mapped to the trivial ${}^{t_1}G''$-torsor. Then, there exists $t_5 \in H^1(U_{\mathrm{ft}}, G')$ mapping to $\theta_{t_4}(t_0)$. Nevertheless, $t_5$ extends across to a torsor $t_6 \in H^1(X_{\mathrm{ft}}, G')$, which maps to a torsor $t_7 \in \check{H}^1(X_{\mathrm{ft}}, {}^{t_3}G)$, which by commutativity restricts to $\theta_{t_4}(t_0)$. However, we also have the commutative square

$$\begin{array}{ccc} \check{H}^1(X_{\mathrm{ft}}, G) & \xrightarrow{\theta_{t_3}} & \check{H}^1(X_{\mathrm{ft}}, {}^{t_3}G) \\ \downarrow & & \downarrow \\ \check{H}^1(U_{\mathrm{ft}}, G) & \xrightarrow{\theta_{t_4}} & \check{H}^1(U_{\mathrm{ft}}, {}^{t_4}G) \end{array}$$

from which it is clear that the unique $t_8 \in \check{H}^1(X_{\mathrm{ft}}, G)$ such that $\theta_{t_3}(t_8) = t_7$ restricts to $t_0$, *i.e.* $\varrho_X^1(G)(t_8) = t_0$, as desired.

*Acknowledgement.* The author is deeply thankful to Bhargav Bhatt who taught him the use of twisted forms to control the fibers of the map $\check{H}^1(Y_{\mathrm{ft}}, G) \to \check{H}^1(Y_{\mathrm{ft}}, G/H)$, where $H$ is a normal subgroup of $G$.

## 5.3 The case of linearly reductive torsors

Remember that linearly reductive group-schemes are extensions of étale groups whose order is prime to $p$, by a connected group-scheme $D(\Gamma)$, where $o(\Gamma)$ is a power of $p$. In view of this, we will focus on $\boldsymbol{\mu}_q$.

By Kummer Theory, see [Mil80, Chapter III, §4],[9] $\boldsymbol{\mu}_n$-torsors over $U$ are in one-to-one correspondence with pairs $(\mathscr{L}, \varphi)$, where $\mathscr{L}$ is an invertible sheaf on $U$ and $\varphi$

[9] From analyzing the long exact sequence on flat cohomology derived from (2.1).

an isomorphism $\mathscr{O}_U \to \mathscr{L}^n$. In fact, the $\boldsymbol{\mu}_n$-torsor, say $V \to U$, associated to the pair $(\mathscr{L}, \varphi)$ is the cyclic cover $\mathbf{Spec} \bigoplus_{i=0}^{n-1} \mathscr{L}^i$. More precisely, $V$ is the open subscheme lying over $U$ of the spectrum of the following semi-local $R$-algebra $C = C(\mathscr{L}, \varphi)$: as an $R$-module $C(\mathscr{L}, \varphi)$ is given by

$$C(\mathscr{L}, \varphi) := \bigoplus_{i=0}^{n-1} H^0(U, \mathscr{L}^i)$$

the multiplication is given by the canonical $R$-linear maps:

$$\begin{aligned} &H^0(U, \mathscr{L}^i) \otimes_R H^0(U, \mathscr{L}^j) \to H^0(U, \mathscr{L}^{i+j}) && \text{if } i+j < n, \\ &H^0(U, \mathscr{L}^i) \otimes_R H^0(U, \mathscr{L}^j) \to H^0(U, \mathscr{L}^{i+j-n}) && \text{if } i+j \geq n. \end{aligned}$$

The coaction $C \to C \otimes \mathscr{O}(\boldsymbol{\mu}_n)$ is given by sending $f \in H^0(U, \mathscr{L}^i)$ to $f \otimes t^i$. Therefore, the trace $\mathrm{Tr}_{C/R}$ is the projection onto the zeroth-degree component of the above direct sum.

For sake of concreteness, we shall realize $\mathscr{L}$ as a subsheaf of $\mathscr{K}(U)$, the sheaf of rational functions on $U$, this by taking a global section $\mathscr{O}_U \to \mathscr{L}$. That is, we may replace $\mathscr{L}$ by $\mathscr{O}_U(D)$, for some Cartier divisor $D$ on $U$. Since $Z$ has codimension at least 2, $D$ extends uniquely to a Weil divisor on $X$; we do not distinguish notationally between them though. Thus,

$$H^0(U, \mathscr{L}^i) = H^0\big(U, \mathscr{O}_U(iD)\big) = R(iD) = \{f \in K \mid \operatorname{div}(f) + iD \geq 0\}.$$

Moreover, an isomorphism $\varphi\colon \mathscr{O}_U \to \mathscr{O}_U(nD)$ amounts to give $a \in K^\times$ such that $\operatorname{div}(a) + nD = 0$, which implies $R(nD) = R \cdot a \subset K$. Hence, we can also present the data of a cyclic cover $C = C(\mathscr{L}, \varphi)$ as $C = C(D; a, n)$. In this way, the product is performed internally by the pairing

$$R(iD) \otimes_R R(jD) \to R\big((i+j)D\big), \quad f \otimes g \mapsto f \cdot g$$

if $i + j < n$. In case $m := i + j - n \geq 0$, we must utilize the following isomorphisms

$$\begin{aligned} R(iD) \otimes_R R(jD) \to R\big((i+j)D\big) = R(mD + nD) &\xleftarrow{\cong} R(mD) \otimes_R R(nD) \\ &= R(mD) \otimes_R R \cdot a \xleftarrow{\cong} R\big(mD\big) \end{aligned}$$

where the first isomorphism "$\xleftarrow{\cong}$" follows from the fact that $nD$ is Cartier. Succinctly, if $i + j \geq n$, the pairing is given by

$$R(iD) \otimes_R R(jD) \to R\big((i+j-n)D\big), \quad f \otimes g \mapsto fg/a.$$

We would like to know in case $C$ is local what its maximal ideal looks like. For this,

**Proposition 5.3.1.** *If $n$ equals the index of $D$, then $C$ is local domain and its maximal ideal is*

$$\mathfrak{n}_C = \mathfrak{m} \oplus \bigoplus_{i=1}^{n-1} R(iD)$$

*In particular, Question 5.1.18 has an affirmative answer, whereby $(C, \mathfrak{n}_C, \mathscr{k})$ is strongly $F$-regular with $F$-signature $s(C) = n \cdot s(R)$, or only $F$-pure if $R$ were only assumed $F$-pure.*

*Proof.* To see $C$ is local, it suffices to show the $R$-submodule $\mathfrak{m} \oplus \bigoplus_{i=1}^{n-1} R(iD)$ of $C$ is an ideal. For this, it is enough to prove that if $f \in R(iD)$, $g \in R(jD)$ with $i, j > 0$ and $i + j = n$, then $fg/a \in \mathfrak{m}$.

If $fg/a \notin \mathfrak{m}$, then $fg = ua$ for some unit $u \in R^{\times}$. We claim this implies $\mathrm{div}(f) = iD$ and $\mathrm{div}(g) = jD$, contradicting $n$ is the index of $D$.

To prove the equalities $\mathrm{div}(f) = iD$ and $\mathrm{div}(g) = jD$, we notice these can be checked locally at every height-1 prime ideal of $R$, then for this we reduce to the case $R$ is a DVR, with valuation val. Let $m$ be the coefficient of $D$ at $\mathfrak{m}$ and say $\mathrm{val}(f) = im + \varepsilon$, $\mathrm{val}(g) = jm + \delta$ for some integers $\varepsilon$, $\delta \geq 0$. Then,

$$nm = \mathrm{val}(a) = \mathrm{val}(fg) = (i + j)m + \varepsilon + \delta = nm + \varepsilon + \delta$$

which forces $\varepsilon$ and $\delta$ to be zero, as required.

The rest is a direct consequence of Theorem 3.0.1, including that $C$ is a domain Corollary 3.0.4.[10] ☕

*Remark* 5.3.2. Notice $(C, \mathfrak{n}_C, \mathscr{k})$ is necessarily strictly local. Indeed, any finite local extension of a Henselian local ring is Henselian; see [Mil80, Chapter I, §4, Corollary 4.3].

*Terminology* 5.3.3. Observe that $R \subset C$ is a torsor (everywhere) if and only if $D$ is Cartier on $X$. In such a case, one says the cyclic cover is of *Kummer-type*. Otherwise, if $n$ is the index of $D$, one says the cyclic cover is of *Veronese-type*.

[10] An alternate, neat proof that $C$ is a domain can be found in [TW92, Corollary 1.9]. In fact, one proves the fraction field of $C$ is $L = K[t]/(t^n - a)$, and one shows $L$ is a field as a consequence of the minimality of $n$.

We learned from Example 5.1.16 that Kummer-type covers may be a source of problems for us, whereas from Proposition 5.3.1, we know Veronese-type cyclic covers are suitable for our purposes. For a general cyclic cover, let $n = m^e \cdot n'$, where $m$ is the index of $D$, $m \nmid n'$ and $e \geq 1$. Hence, $n'D =: D'$ has index $m$ too, so that we have the decomposition

$$R \subset C' := C(n'D; a, m^e) \subset C$$

where $R \subset C'$ is a $\boldsymbol{\mu}_{m^e}$-torsor away from $Z$. We can filter further as

$$R \subset C'' := C(mn'D; a, m^{e-1}) \subset C'.$$

But now $mn'D$ is Cartier, so that $C''$ is a Kummer-type extension of $R$, that is

$$C'' \cong R[t]/\big(t^{m^{e-1}} - u\big), \quad u \in R^\times$$

Indeed, if $mn'D + \operatorname{div}(b) = 0$, then $\operatorname{div}\big(b^{m^{e-1}}\big) = \operatorname{div}(a)$, which means $b^{m^{e-1}} = ua$ for some unit $u$ of $R$. Moreover, $C'$ can be written as follows:

$$C' = \bigoplus_{i=1}^{m-1} C'' \otimes_R R(iD') = C'' \oplus \big(C'' \otimes_R R(D')\big) \oplus \cdots \oplus \big(C'' \otimes_R R\big((m-1)D'\big)\big).$$

If $e = 1$, we need not deal with $C''$. Otherwise, assuming $C$ is local and remembering $\mathscr{k}$ is perfect, we have that $C''$ has to be local with residue field $\mathscr{k}$, which implies $m = q$ a power of $p$ and $u = v^{q^{e-1}} + x$ for some $v \in R^\times$ and $x \in \mathfrak{m}$. If $x \in R^{q^{e-1}}$, $C''$ is a trivial torsor and we could get rid of it, otherwise we saw in Example 5.1.16 how $C''$ may not be strongly $F$-regular, and so we cannot expect to have an affirmative answer for Question 5.1.18 for general cyclic covers, unless for instance $n = p$ and we assume the closed points are $\mathscr{k}$-rational. Nonetheless, we at least have the following.

**Lemma 5.3.4.** *As in Question 5.1.18, let $(R, \mathfrak{m}, \mathscr{k}) \subset (S, \mathfrak{n}, \mathscr{k})$ a $\boldsymbol{\mu}_n$-quotient that is a torsor in codimension 1 but not everywhere. Then there exists a nontrivial local extension $(R, \mathfrak{m}, \mathscr{k}) \subset (S', \mathfrak{n}', \mathscr{k})$, which is of Veronese-type over $(R, \mathfrak{m})$. In particular, $(S', \mathfrak{n}', \mathscr{k})$ is a strongly $F$-regular $\mathscr{k}$-rational germ.*

*Proof.* Let $D$ be the divisor associated to the cyclic cover $R \subset S$; we know by hypothesis its index is not 1, for otherwise, $R \subset S$ would be a $\boldsymbol{\mu}_n$torsor everywhere. Thus, $S'$ can be taken to be $C(D, m)$, where $m$ is the index of $D$. For the last assertion, see Proposition 5.3.1 and Remark 5.3.2. ☕

## 5.4 On the existence of a maximal cover

We are ready to state our main results. Recall that we assume $(R, \mathfrak{m}, \Bbbk, K)$ to be a strongly $F$-regular $\Bbbk$-rational germ over an algebraically closed field $\Bbbk$. We denote by $X$ the spectrum of $R$, and by $Z \subset X$ a fixed closed subscheme of codimension at least 2 and ideal of definition $I \subset R$.

**Theorem 5.4.1.** *Let $(R, \mathfrak{m}, \Bbbk) \subset (S, \mathfrak{n}, \Bbbk)$ be a $G$-torsor away from $Z$ but not everywhere, with $G^\circ$ abelian. There exists a nontrivial linearly reductive subgroup-scheme $G' \subset G$ and $(R, \mathfrak{m}, \Bbbk) \subset (S', \mathfrak{n}', \Bbbk)$ a $G'$-torsor away from $Z$, such that $(S', \mathfrak{n}')$ is a strongly $F$-regular $\Bbbk$-rational germ with $s(S') = o(G') \cdot s(R)$. In particular, $o(G') \leq 1/s(R)$.*

*Proof.* Using that $G = G^\circ \rtimes \pi_0(G)$, we may decompose the given local extension $(R, \mathfrak{m}) \subset (S, \mathfrak{n})$ as

$$(R, \mathfrak{m}) \subset \left(S^{G^\circ}, \mathfrak{n} \cap S^{G^\circ}\right) \subset (S, \mathfrak{n})$$

where the former extension is a $\pi_0(G)$-torsor away from $Z$. If $(R, \mathfrak{m}) \subset \left(S^{G^\circ}, \mathfrak{n} \cap S^{G^\circ}\right)$ is proper, we are done, for in that case, $p \nmid o\big(\pi_0(G)\big)$ by Proposition 3.0.18, and so $\pi_0(G)$ is linearly reductive, by Nagata's theorem [Nag62].

In this manner, we may assume $G$ is connected, and further abelian connected by hypothesis. Therefore, $G$ is the trivial extension of a multiplicative type group-scheme by a unipotent one. Then, by Lemma 5.2.3 and Theorem 5.2.1, we may further assume $G$ is of multiplicative type, as well as connected. Hence, we may assume $G$ isomorphic to a finite direct sum $\bigoplus_i \boldsymbol{\mu}_{p^{e_i}}$. Using Lemma 5.2.3 once again , we may further assume $G \cong \boldsymbol{\mu}_{p^e}$. In this way, the result follows directly from Lemma 5.3.4 and Proposition 5.3.1. ☕

By iteration of Theorem 5.4.1, we obtain our main result on the existence of a maximal cover.

**Theorem 5.4.2** (Main result: existence of a maximal cover)**.** *For any strongly $F$-regular $\Bbbk$-rational germ $(R, \mathfrak{m}, \Bbbk, K)$, there exists a chain of finite extensions of strongly $F$-regular strictly local domains*

$$(R, \mathfrak{m}, \Bbbk, K) \subsetneq (S_1, \mathfrak{n}_1, \Bbbk, L_1) \subsetneq \cdots \subsetneq (L_t, \mathfrak{n}_t, \Bbbk, L_t) = (S^\star, \mathfrak{n}^\star, \Bbbk, L^\star)$$

*where each intermediate extension $(S_i, \mathfrak{n}_i, \Bbbk) \subsetneq (S_{i+1}, \mathfrak{n}_{i+1}, \Bbbk)$ is a $G_i$-torsor away from $IS_i$, such that: each $G_i$ is a linearly reductive group-scheme, $[L^\star : K]$ is at most $1/s(R)$, and any further $G$-torsor over $(S^\star, \mathfrak{n}^\star)$ away from $IS^\star$, such that $G^\circ$ is either trigonalizable or nilpotent, is a torsor everywhere.*

*Proof.* By iterating Theorem 5.4.1 until $s(R)$ is exhausted, and using our preliminary discussion in Proposition 2.3.5, we obtain the result for the case of abelian connected components at the identity. It remains to prove it for both trigonalizable and nilpotent connected components at the identity. As in the proof of Theorem 5.4.1, we may assume $G$ is connected and so either trigonalizable or nilpotent. Hence, it suffices to prove $\varrho_{X^\star}(G)\colon \check{H}^1(X^\star_{\mathrm{ft}}, G) \to \check{H}^1(U^\star_{\mathrm{ft}}, G)$ is surjective for all trigonalizable and all nilpotent group-schemes $G$.

If $G$ is trigonalizable, it has a nontrivial normal subgroup $G'$ that is isomorphic to either $\boldsymbol{\alpha}_p$ or $(\mathbb{Z}/p\mathbb{Z})^{\oplus m}$ for some $m$; see [Mil15, Corollary 17.25]. If $G' = G$, we are done, otherwise we may proceed by induction on the order as $G'' := G/G'$ is trigonalizable.

If $G$ is nilpotent, it has a nontrivial central subgroup $G'$ (so normal and abelian) whose quotient is nilpotent. If $G = G'$, we are done, otherwise we may proceed by induction on the order.

In either case, the proofs are *verbatim* the same as our second proof of Theorem 5.2.1. The only difference is explaining why the twisted form of $G'$ by a $G$-torsor is trivial. In the proof of Theorem 5.2.1, we had this because $G'$ was taken to be central. Hence, the same works for the nilpotent case. The trigonalizable case requires a different explanation though. The following claim does the job, using $Y$ equal to $X^\star$ or $U^\star$.

**Claim 5.4.3.** *If $Y$ is a scheme with torsion-free Picard group and trivial étale fundamental group, then the map $\check{H}^1(Y_{\mathrm{ft}}, G) \to \check{H}^1(Y_{\mathrm{ft}}, \operatorname{Aut} G')$ induced by $G \to \operatorname{Aut} G'$ is trivial if either $G' = \boldsymbol{\alpha}_p$ or if $\operatorname{Aut} G'$ is finite étale, e.g. $G' = (\mathbb{Z}/p\mathbb{Z})^{\oplus m}$.*

*Proof of claim.* Notice that $\check{H}^1(Y_{\mathrm{ft}}, \operatorname{Aut} G') = 0$ if $\operatorname{Aut} G'$ is finite étale; by the assumption on the étale fundamental group of $Y$, this case is then trivial. On the other hand, observe that $\operatorname{Aut} \boldsymbol{\alpha}_p = \mathbb{G}_{\mathrm{m}}$; therefore, $\check{H}^1(Y_{\mathrm{ft}}, \operatorname{Aut} \boldsymbol{\alpha}_p) = \operatorname{Pic} Y$. Since $G$ is finite, the map $\check{H}^1(Y_{\mathrm{ft}}, G) \to \operatorname{Pic} Y$ sends $G$-torsors to torsion line bundles, which are trivial by assumption. This proves the claim. ☕

This proves the theorem. ☕

*Remark* 5.4.4 (The relation with Esnault and Viehweg's local Nori fundamental group-scheme). It is natural to ask what the relationship is between our maximal cover $R \subset S^{\star}$ and Esnault and Viehweg's construction of the local Nori fundamental group-scheme in [EV10]. To simplify our discussion, let us consider abelian group-schemes only. In this case, our cover has trivial abelian local Nori fundamental group-scheme $\pi_{1,\mathrm{loc}}^{\mathrm{N,ab}}(U^{\star}, X^{\star}, x^{\star})$. Nevertheless, it is not clear to the author whether or not this implies that $\pi_{1,\mathrm{loc}}^{\mathrm{N,ab}}(U, X, x)$ is finite. We are deeply thankful to Christian Liedtke who pointed out to us an example suggesting this should not be always the case. He kindly shared with us the following example of surface $\mathrm{D}_4$ singularity in characteristic 2, which is a $\mathbb{Z}/2\mathbb{Z}$-quotient of $\hat{\mathbb{A}}^2_{\mathbb{k}}$ but admits a nontrivial $\boldsymbol{\mu}_2$-torsor. The singularity is $\mathbb{k}[\![x, y, x]\!]/(z^2 - xyz - x^2y - xy^2)$; the Weil divisor corresponding to the prime ideal $(x, z)$ has index 2. Therefore, its local abelian Nori fundamental group-scheme cannot be $\mathbb{Z}/2\mathbb{Z}$ since it needs to take into account $\boldsymbol{\mu}_2$. Nonetheless, this singularity, though $F$-pure, is not strongly $F$-regular. We invite the reader to look at [ST14, Example 7.12] for a closer look into this particular singularity. In fact, this was the same example we used to proof the necessity of the surjectivity of $T$ in Example 3.0.15.

# Chapter 6

# Additional corollaries of the transformation rule

In the previous two chapters, we learned how to use the transformation rule for the $F$-signature under finite covers to study finite torsors over strongly $F$-regular singularities, the main topic of this dissertation. In fact, it was in that context that this transformation rule was first discovered and motivated. The goal in this chapter is to give additional consequences of this transformation rule that are interesting on their own.

## 6.1 Purity of the Branch Locus

We start with what is perhaps the most fundamental of these additional corollaries of the transformation rule. This is a new purity of the branch locus theorem for mild singularities in positive characteristic. By mild we mean singularities with $F$-signature larger than $1/2$. In particular, we give a new proof for the classical Zariski–Nagata–Auslander purity of the branch locus theorem for regular schemes, [Zar58, Nag58, Aus62].

As a historical aside, it is worth mentioning that $E$. Kunz noticed in [Kun69b] that if a local ring $R$ is such that $F^e_*R$ is flat over $R$ for some (then all) $e \in \mathbb{N}$, then $R$ satisfies purity of the branch locus. However, in a footnote, he mentioned that in fact, this condition characterizes regularity in positive characteristic, and the forthcoming paper [Kun69a] would contain a proof of this. In this way, we can see how seeking for a simple proof for the purity of the branch locus inspired Kunz's fundamental characterization of regularity in positive characteristic by the flatness of the Frobenius endomorphism. Our purity result can then be thought as a generalization of his, for ours roughly says that if the $R$-modules $F^e_*R$ are asymptotically more than "half free," then purity of the branch locus holds.

It is worth comparing with other results about purity for singular rings [Gro63, Cut95]. These results are, however, of a different nature. For example, Grothendieck proved purity of the branch locus for complete intersection of dimension $\geq 3$. On the other hand, Cutkosky's purity result is an improvement of the one for complete intersections.

Before establishing our purity result, we would like to give separately the key ingredient for purity. Roughly speaking, it says that the $F$-signature goes up under the presence of ramification.

**Corollary 6.1.1** (The $F$-signature goes up under the presence of ramification)**.** *Suppose that $(R, \mathfrak{m}, \mathscr{k}, K) \subset (S, \mathfrak{n}, \mathscr{l}, L)$ is a local extension of normal domains of dimension $d \geq 2$. If the cover $f \colon \operatorname{Spec} S \to \operatorname{Spec} R$ is quasi-étale but not étale, then we have that $s(S, f^*\Delta) \geq 2 \cdot s(R, \Delta)$ for all $\mathbb{Q}$-divisor $\Delta$ on $\operatorname{Spec} R$.*

*Proof.* We may assume without lost of generality that $(R, \Delta)$ is strongly $F$-regular (in particular $F$-pure), otherwise the statement is trivial.

Now, the transformation rule Theorem 3.0.1 gives that $s(S) = n \cdot s(R)$ with $n = [L : K]/[\mathscr{l} : \mathscr{k}]$. By Proposition 3.0.17, $n$ is an integer. It remains to prove $n \neq 1$. However, if $[L : K] = [\mathscr{l} : \mathscr{k}]$, then the extension is free; see [Har77, Chapter II, Lemma 2.8]. Hence, it is étale everywhere, by purity of faithfully flat morphisms [AK70, Chapter VI, Theorem 6.8]. ☕

In this way, we have that purity for the branch locus holds for singularities so that $s(R, \Delta) > 1/2$.

**Corollary 6.1.2** (Purity of the branch locus for mild singularities)**.** *Let $(R, \mathfrak{m}, \mathscr{k}, \Delta)$ be a local ring with $s(R, \Delta) > 1/2$, then any quasi-étale cover $Y \to \operatorname{Spec} R$ is étale.*

*Proof.* We may assume without lost of generality that $\Delta = 0$, for $s(R) \geq s(R, \Delta)$. Let $Y = \operatorname{Spec} S$ and suppose for sake of contradiction that $R \subset S$ is not étale. By tensoring $R \subset S$ by $R^{\mathrm{h}}_{\mathfrak{m}}$, the henselization of $R$ at $\mathfrak{m}$, we get a finite extension $R^{\mathrm{h}}_{\mathfrak{m}} \subset R^{\mathrm{h}}_{\mathfrak{m}} \otimes_R S = S_1 \times \cdots \times S_n$ where the rings $S_i$ are local. Now, since the (quasi-étale) extension $R^{\mathrm{h}}_{\mathfrak{m}} \subset S_1 \times \cdots S_n$ is not étale, at least one of the finite extensions $R^{\mathrm{h}}_{\mathfrak{m}} \subset S_i$ is not étale (but quasi-étale). Therefore,

$$1 \geq s(S_i) \geq 2 \cdot s\big(R^{\mathrm{h}}_{\mathfrak{m}}\big) = 2 \cdot s(R) > 2 \cdot \frac{1}{2} = 1$$

which is a contradiction. Indeed, $s\big(R^{\mathrm{h}}_{\mathfrak{m}}\big) = s(R)$ because $R \to R^{\mathrm{h}}_{\mathfrak{m}}$ is flat with regular closed fiber; see [Yao06, Theorem 5.6], [CST17, Theorem 3.5]. ☕

## 6.2 Effective upper bounds on divisor torsion

In this section, we bound the torsion of the divisor class group of a strongly $F$-regular local ring, and a globally $F$-regular projective variety. We begin with the local part.

**Corollary 6.2.1.** *Let $(R, \mathfrak{m}, \mathscr{k})$ be a local $F$-regular ring with $Z \subset X = \operatorname{Spec} R$ a closed subscheme of codimension at least 2 and open complement $U$. The torsion of the Picard group $\operatorname{Pic} U = \operatorname{Cl} R$ is bounded by $1/s(R)$, i.e. if $\mathscr{L} \in \operatorname{Pic} U$ has index $n$, then $n \leq 1/s(R)$. In particular, if $s(R) > 1/2$, then $\operatorname{Pic} U$ is torsion-free.*

*Proof.* Let $\mathscr{L} \in \operatorname{Pic} U$ with index $n$, and say $\varphi \colon \mathscr{O}_U \to \mathscr{L}^n$ is an isomorphism. Then by Proposition 5.3.1, we have that $n \cdot s(R) = s\big(C(\mathscr{L}, \varphi)\big) \leq 1$. ☕

**Corollary 6.2.2.** *Let $\mathscr{A}$ be an ample line bundle on a globally $F$-regular projective variety $Y$ over $\mathscr{k} = \mathscr{k}^{\mathrm{alg}}$, and let $A := \bigoplus_{i \geq 0} H^0(Y, \mathscr{A}^i)$ be the associated section ring of $Y$. If $\mathscr{A} = \mathscr{L}^n$ for another line bundle $\mathscr{L}$, then $n \leq 1/s(A_O)$, where $O \in \operatorname{Spec} A =: C(Y)$ is the vertex point of the affine cone $C(Y)$.*

*Proof.* First of all, remember $Y$ is globally $F$-regular if and only if $A$ is strongly $F$-regular, by the work [SS10]. In particular, $s(A_O) > 0$. Let $B := \bigoplus_{i \geq 0} H^0(Y, \mathscr{L}^i)$. Then $A$ is the $n$-th Veronese subring of $B$. By taking (strict) Henselizations at the origin, we get an inclusion $A^{\mathrm{sh}}_O \subset B^{\mathrm{sh}}_O$, which is cyclic of Veronese-type of index $n$. Notice $A^{\mathrm{sh}}_O$ is a strongly $F$-regular $\mathscr{k}$-rational germ of $F$-signature $s(A_O)$.[1] Therefore, $1 \geq s(B^{\mathrm{sh}}_O) = n \cdot s(A)$. ☕

The global part then is:

**Corollary 6.2.3.** *With the same setup as in Corollary 6.2.2, the torsion of $\operatorname{Cl} Y$, the divisor class group of $Y$, is bounded by $1/s(A_O)$.*

*Proof.* We simply apply [Har77, Chapter II, Exercise 6.3]. Thus, we have a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{u} \operatorname{Cl} Y \xrightarrow{v} \operatorname{Cl}\big(C(Y)\big) \to 0.$$

[1]For, $A_O \to A^{\mathrm{sh}}_O$ is flat with regular closed fiber, then use [Yao06, Theorem 5.6], [CST17, Theorem 3.5].

On the other hand, $\mathrm{Cl}\big(C(Y)\big) \cong \mathrm{Cl}\big(\operatorname{Spec} A_O\big) \cong \operatorname{Pic} U$, where $U$ is the regular locus of $\operatorname{Spec} A_O$. Therefore, by Corollary 6.2.1, the torsion of this group is bounded by $1/s(A_O)$.

These two facts together imply the torsion of $\mathrm{Cl}\,Y$ is bounded by $1/s(A_O)$. Indeed, let $D \in \mathrm{Cl}\,Y$ of order $n$. Then, $nv(D) = v(nD) = 0$, so we have that the index of $v(D)$, say $m \leq 1/s(A_O)$, divides $n$. In particular, $mD$ belongs to the kernel of $v$, so there is $l \in \mathbb{Z}$ such that $u(l) = mD$, but then $(n/m) \cdot u(l) = (n/m)mD = nD = 0$, *i.e.* $u((n/m)l) = 0$, which implies $l = 0$. Hence, $mD = 0$, so $n \mid m$, therefore $n = m$ and $n \leq s(A_O)$ as desired. ☕

*Remark* 6.2.4. Since $Y$ in Corollary 6.2.3 is normal, we have $\operatorname{Pic} Y \subset \mathrm{Cl}\,Y$. Therefore, this result also bounds the torsion of the Picard group of $Y$.

Recall that globally $F$-regular varieties are an analog of log-Fano varieties in characteristic zero; see [SS10] for full details on this analogy. It is in this regard that Corollary 6.2.3 should be compared to [Xu14, Proposition 1]. Concretely, if $D$ is an order $n$ Weil divisor[2] on $(Y, \Delta)$, a log-Fano pair, then the corresponding Veronese-type cyclic cover $Y' \to Y$ defines a finite surjective morphism of degree $n$. Therefore, as a direct application of [Xu14, Proposition 1] and its proof, we have

$$n \leq C/\operatorname{vol}(K_Y + \Delta)$$

where $C = C(r, \dim Y)$ is a constant depending only on $r$ the index of $K_Y + \Delta$ (as $\mathbb{Q}$-Cartier divisor) and the dimension of $Y$. This constant is shown to exist in the fundamental paper [HMX14, Corollary 1.8]. Then, it is natural to ask:

*Question* 6.2.5. What is the relation between the numbers $1/s(A_O)$ and $C/\operatorname{vol}(K_Y + \Delta)$?

## 6.3 Veronese-type cyclic covers of $F$-singularities

Let $(R, \mathfrak{m}, \mathscr{k}, K)$ be a $\mathbb{Q}$-Gorenstein local domain with canonical divisor $K_R$ of index $n$. Then, the corresponding Veronese-type cyclic cover $C(K_R; n)$ is called a *canonical* cover of $(R, \mathfrak{m})$. As explained in Section 5.3, the Veronese-type cyclic cover associated

[2] Meaning $nD \sim 0$ and $n$ is the smallest positive integer with this property.

to a divisor $D$ on $\operatorname{Spec} R$ of index $n$ is given by the local ring

$$C = C(D; n) = \bigoplus_{i=0}^{n-1} R(iD)$$

with maximal ideal $\mathfrak{n}_C = \mathfrak{m} \oplus \bigoplus_{i=1}^{n-1} R(iD)$, and residue field $\mathscr{k}$.

In [Wat91], K.-i. Watanabe proved that if $n$ is prime-to-$p$, then $C(K_R; n)$ is strongly $F$-regular (resp. $F$-pure) if $R$ is strongly $F$-regular (resp. $F$-pure). In the following corollary, we proved this is the case even if $p$ divides $n$ for all Veronese-type cyclic covers.

**Corollary 6.3.1** (Strong $F$-regularity and $F$-purity transfer to Veronese-type cyclic covers)**.** *Let $(R, \mathfrak{m}, \mathscr{k}, K)$ be a normal local domain of dimension at least* 2*. For any $\mathbb{Q}$-Cartier divisor $D$ on* $\operatorname{Spec} R$ *of index $n$, we have that $C(D; n)$ is strongly $F$-regular (resp. $F$-pure) if and only if $R$ is strongly $F$-regular (resp. $F$-pure), even if $p$ divides $n$. Moreover, $s(C) = n \cdot s(R)$.*

*Proof.* We have that $(R, \mathfrak{m}, \mathscr{k}) \subset (C, \mathfrak{n}_C, \mathscr{k})$ is a $\boldsymbol{\mu}_n$-torsor in codimension 1; therefore, the transformation rule Theorem 3.0.1 applies with $T = \operatorname{Tr}_{C/R}$ equal to the projection onto the zeroth-degree direct summand. One then applies Corollary 3.0.4 and Scholium 3.0.9. ☕

*Remark* 6.3.2. It is worth remarking that $F$-rationality (nor $F$-injectivity) are not necessarily transfered to Veronese-type cyclic covers. Indeed, A. Singh provided counterexamples for this in [Sin03].

Moreover, we also obtain effective upper bounds on the index of $\mathbb{Q}$-Cartier divisors and a simultaneous index-one cover result as in [GKP16, Theorem 1.10].

**Corollary 6.3.3** (Simultaneous index-one cover)**.** *Let $R \subset S^\star$ be as in Theorem 5.4.2 with $I \subset R$ cutting out the singular locus of $X =$* $\operatorname{Spec} R$*, and let $h \colon Y^\star \to X$ be the induced morphism. If $D$ is a $\mathbb{Q}$-divisor Cartier divisor on $X$ of index $n$, then $n \leq 1/s(R)$. Moreover, $h^*D$ is a Cartier divisor on $Y^\star$.*

*Proof.* The bound $n \geq 1/s(R)$ follows at once from Corollary 6.3.1. Observe $h^*D$ must be Cartier because if its index is not 1, then its corresponding Veronese-type cyclic cover is a cover over $Y^\star$ violating its maximality. ☕

# Chapter 7

# Further developments and questions

In this final chapter, we conclude by briefly summarizing some recent, interesting research that has arisen from some of the results treated in this dissertation. Also, we discuss some additional questions originated from our results.

## 7.1 Global aspects

In [GKP16], Greb–Kebekus–Peternell used Xu's local result [Xu14] to prove that for a quasi-projective variety $X/\mathbb{C}$ with KLT singularities, any sequence of finite quasi-étale and generically Galois covers eventually becomes étale. Equivalently, for any quasi-projective KLT complex variety, there exists a finite quasi-étale generically Galois cover $h\colon Y^\star \to X$ such that any further quasi-étale cover $Y \to Y^\star$ is actually étale. Their proof was essentially based on using a Whitney stratification to bootstrap Xu's local result globally.

Inspired by the aforementioned local-global interplay, in [BCRG$^+$17], the author, in collaboration with B. Bhatt, P. Graff, K. Schwede, and K. Tucker, studied the ramifications of Theorem 4.0.1 to the global properties of spaces with strongly $F$-regular singularities. In this paper, building on Theorem 4.0.1, a stratification theorem of O. Gabber [ILO14, Exposé XXI, Théorème 1.1, 1.3], and recent work on the non-local behavior of $F$-signature [DPY16], we proved the analog of the main result of Greb–Kebekus–Peternell [GKP16] in characteristic $p > 0$.

**Theorem 7.1.1** ([BCRG$^+$17])**.** *Suppose $X$ is an $F$-finite Noetherian integral scheme with only strongly $F$-regular singularities. Suppose we are given a sequence of finite quasi-étale covers of normal integral schemes*

$$X \xleftarrow{\gamma_1} Y_1 \xleftarrow{\gamma_2} Y_2 \xleftarrow{\gamma_3} \cdots$$

*such that each $X_i/X$ is generically Galois. Then, all but finitely many $\gamma_i$ are étale. In particular, there exists a finite quasi-étale generically Galois cover $Y^\star \to X$ so that any further finite quasi-étale cover $Y \to Y^\star$ is étale. Equivalently, the map $\pi_1^{\text{ét}}(Y^\star_{\text{reg}}) \to \pi_1^{\text{ét}}(Y^\star)$ induced by the inclusion of the regular locus is an isomorphism.*

We would like to point out that a comprehensive treatment of the main result in both [GKP16] and [BCRG$^+$17] have been given in [Sti17] by C. Stibitz. He used de Jong's alterations [dJ96] to find a common replacement for the Whitney stratification in the complex case and Gabber's constructibility in positive characteristic.

It is then natural to pose the following question.

*Question* 7.1.2. What are the global consequences of bootstrapping our local results in Chapter 5 globally? More concretelly, given a scheme $X$ with only strongly $F$-regular singularities, does there exist a suitably nice cover $Y^\star \to X$ such that every abelian (trigonalizable, nilpotent) finite torsor over the regular locus of $Y^\star$ is a torsor everywhere?

More specifically, we can always formulate the following question.

*Question* 7.1.3. Is it possible to bootstrap Corollary 6.3.3 to obtain a global index for $\mathbb{Q}$-Cartier divisors on an $F$-finite Noetherian $\Bbbk$-scheme with strongly $F$-regular singularities as in [GKP16, Remark 1.11], and [BCRG$^+$17, Corollary 4.8]? That is, is it possible to use Corollary 6.3.3 to obtain an $N > 0$ as in [BCRG$^+$17, Corollary 4.8 (b)] that works for all $\mathbb{Q}$-divisors and not only for $\mathbb{Z}_{(p)}$-divisors?

## 7.2 Connection to characteristic zero

Let $X/\Bbbk$ be a finite type $\Bbbk$-scheme over an algebraically closed field $\Bbbk$ and let $x \in X(\Bbbk)$ be a $\Bbbk$-rational closed point, say $\dim X \geq 2$. To simplify notation, let $\pi_1^{\text{loc}}(X, x)$ be the étale fundamental group of $\operatorname{Spec}(\hat{\mathscr{O}}_{X,x}) \smallsetminus \{x\}$. Then, from Xu's [Xu14, Theorem 1], we know that $\pi_1^{\text{loc}}(X, x)$ is finite if $\Bbbk = \mathbb{C}$ and $X$ has KLT singularities. On the other hand, from Theorem 4.0.1, we know $\pi_1^{\text{loc}}(X, x)$ is finite if $\Bbbk$ has positive characteristic and $X$ has strongly $F$-regular singularities. Beside the similarities between both in results, there is a considerable contrast in their proofs. Namely, Xu's uses fundamental advances in the minimal model program [BCHM10], whereas ours is quite more elementary. Therefore, it is natural to wonder whether one could show Xu's result from ours by spreading out to positive characteristic. Affirmatively, B. Bhatt,

O. Gabber, and M. Olsson [BGO17] introduced a spreading out technique for deducing finiteness results for étale fundamental groups in characteristic zero. Their spreading out technique is the following

**Theorem 7.2.1** ([BGO17]). *With $(X, x)$ as above and $\Bbbk = \mathbb{C}$, spread out the pair $(X, x)$ to $(X_A, x_A)$ over a finite type $\mathbb{Z}$-algebra $A \subset \mathbb{C}$. For a geometric point $\bar{y} \to \operatorname{Spec} A$, denote by $(X_{\bar{y}}, x_{\bar{y}})$ the pullback of $(X_A, x_A)$ to $\bar{y}$. If there is $c \in \mathbb{N}$ and a dense open subset $U \subset \operatorname{Spec} A$ such that for all geometric closed point $\bar{y} \to U$, we have*

$$\#\pi_1^{\mathrm{loc}}(X_{\bar{y}}, x_{\bar{y}})^{(p_{\bar{y}})} \leq p_{\bar{y}}^c$$

*where $p_{\bar{y}}$ is the residual characteristic of $\bar{y}$, and $\pi_1^{\mathrm{loc}}(X_{\bar{y}}, x_{\bar{y}})^{(p_{\bar{y}})}$ denotes the maximal prime-to-$p_{\bar{y}}$ quotient of $\pi_1^{\mathrm{loc}}(X_{\bar{y}}, x_{\bar{y}})$. Then, $\pi_1^{\mathrm{loc}}(X, x)$ is finite.*

Then, [Xu14, Theorem 1] follows from Theorem 4.0.1 by proving $s(X_{\bar{y}}, x_{\bar{y}}) \geq 1/p_{\bar{y}}^d$ for some $d$. This is precisely what Bhatt–Gabber–Olsson do in [BGO17, Proposition 6.4].

## 7.3 Beyond trigonalizable and nilpotent torsors

At the moment, it is unclear to the author how to push Theorem 5.4.2 beyond the trigonalizable and nilpotent cases, which are important instances of the solvable case. The general solvable case for example remains open.

Recall that in both of the aforementioned solvable cases, our strategy to understand the surjectivity of $\varrho^1_X(G)\colon \check{H}^1(X_{\mathrm{ft}}, G) \to \check{H}^1(U_{\mathrm{ft}}, G)$ was to analyze it first in the case $G$ is simple abelian to use $H^2(X_{\mathrm{ft}}, G) = 0$, along twisted forms, to extend our knowledge from the simple abelian case to those more general solvable cases.

On the other hand, the author is very grateful to Axel Stäbler who made him aware of the explicit classification of the (connected) simple finite rank-1 group-schemes via the classification of simple restricted Lie algebras in positive characteristic by Block–Wilson–Premet–Strade [BW88], [SW91], [PS08]. We recommend [Viv10] for a very nice, brief account. Then, letting $\varrho^1_X(G)\colon \check{H}^1(X_{\mathrm{ft}}, G) \to \check{H}^1(U_{\mathrm{ft}}, G)$ denote the natural restriction map of $G$-torsors as in Chapter 5, the following questions are of great interest:

*Question* 7.3.1. For which connected simple finite rank-1 group-schemes $G$ is $\varrho^1_X(G)$ onto?

*Question* 7.3.2. If $G$ is a connected finite rank-1 group-scheme so that $\varrho^1_X(G)$ is not surjective, does Question 5.1.18 have an affirmative answer?

*Question* 7.3.3. Given a connected simple finite rank-1 group-scheme $G$, for which type of ($F$-)singularity $X$, if any, is $\varrho^1_X(G)$ naturally surjective?

Last but not least,

*Question* 7.3.4. Is $\check{H}^i(X_{\mathrm{ft}}, G)$ trivial for all finite group-schemes $G$ and all germs $X$?

Based on our experience working in the abelian case, it seems reasonable to expect the answer for Question 7.3.1 to be: for all the Cartan-type ones, and not necessarily for the Classical-type ones. However, for the Classical-type ones, we expect Question 7.3.1 to have an affirmative answer.

## 7.4 Transforming $F$-signature of Cartier algebras

It is natural to ask what is the most general version of our transformation rule for the $F$-signature Theorem 3.0.1. Even better, it is natural to ask what is the most *natural* framework upon which our proof for Theorem 3.0.1, based on duality for finite morphisms, can be expressed. In a work in preparation with A. Stäbler, we seek for that naturality. We prove that a natural framework to describe the transformation rule is the formalism of Cartier algebras and modules, and Blickle-Stäbler's $f^*$ and $f^!$ functors [BS16]. We assume familiarity of the reader with the theory of Cartier algebras [Bli13, Sch11, BB11]. For example, we prove the following transformation rule

**Theorem 7.4.1** (Transformation rule for the $F$-signature of Cartier algebras under finite covers)**.** *Let $(R, \mathfrak{m}, \mathscr{k}, K) \subset (S, \mathfrak{n}, \ell, L)$ be a local finite extension of normal domains, and $f\colon Y \to X$ the induced finite morphisms. Choose a surjective section $T \in \omega_{S/R}$ such that $T(\mathfrak{n}) \subset \mathfrak{m}$. If $\mathscr{C} \subset \mathscr{C}^R$ is a Cartier algebra consisting of $T$-transposable $p^{-1}$-linear maps, then the following formula holds*

$$[\ell : \mathscr{k}] \cdot s(S, f^*\mathscr{C}) = [L : K] \cdot s(R, \mathscr{C})$$

*Moreover, $(S, f^*\mathscr{C})$ is $F$-pure if and only if so is $(R, \mathscr{C})$.*

The last sentence should not be considered as a minor detail. In fact, it explains why we could not get a full converse statement in Scholium 3.0.9. The reason is that

$\mathscr{C}^{\Delta^*}$ is not the pullback of $\mathscr{C}^{\Delta}$, *c.f.* Remark 3.0.8. The perseverence of $F$-purity (and strong $F$-regularity) that is natural to expect is the one between the $F$-purity of the Cartier algebras $\mathscr{C}^{\Delta}$ and $f^*\mathscr{C}^{\Delta}$. In fact, in the same setting as Theorem 7.4.1, we prove that the splitting prime of $(R, \mathscr{C})$ is the contraction of the splitting prime of $(S, f^*\mathscr{C})$. As an application, we also obtain the following transformation rule for splitting ratios

$$\big[k(\mathfrak{n}) : k(\mathfrak{m})\big] \cdot r\big(S, f^*\mathscr{C}\big) = \big[k\big(\wp(f^*\mathscr{C})\big) : k\big(\wp(\mathscr{C})\big)\big] \cdot r(R, \mathscr{C})$$

where $\wp(-)$ denotes the splitting prime of a Cartier algebra and $r(-)$ its splitting ratio.

As an application of this transformation rule, we obtain the following result on tame fundamental groups.

### 7.4.1 Tame fundamental groups

We consider $R$, $X$, and $U$ as in Chapter 5, but now we additionally consider a prime Weil divisor $P$ on $U$, which extends to a unique prime divisor on $X$. We assume that $P$ is a minimal center of $F$-purity [Sch10a]. Then, we consider the Galois category $\mathscr{G}^P(X, U)$ of finite covers over $U$ that are étale away from $P$ but tamely ramified over $P$ [GM71, §2.4], [KS10, §7]. It is of our interest to study the fundamental group $\pi_1^{\mathrm{t},P}(X, U)$ associated to the Galois category $\mathscr{G}^P(X, U)$. For this, we have the following

**Theorem 7.4.2.** *There exists an exact sequence of groups*

$$\hat{\mathbb{Z}}^{(p)} \to \pi_1^{\mathrm{t},P}(X, U) \to G \to 1$$

*where $G$ is a finite group of order at most $1/r\big(R, \mathscr{C}^{[P]}\big)$, and $\hat{\mathbb{Z}}^{(p)}$ is the prime-to-p part of the profinite completion of $\mathbb{Z}$. We denote by $\mathscr{C}^{[P]}$ the Cartier algebra of $P$-compatible $p^{-1}$-linear maps.*

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